Fractional Order Disturbance Observer for Robust Vibration Suppression *

YangQuan Chen † (yqchen@ieee.org)
CSOIS, Utah State University, USA

Blas M. Vinagre (bvinagre@unex.es)
University of Extremadura, Spain

Igor Podlubny (igor.podlubny@tuke.sk)
Technical University of Košice, Slovak Republic

Abstract. For the first time, the fractional order disturbance observer (FO-DOB) is proposed for vibration suppression applications such as hard disk drive servo control. It has been discovered in a recently published US patent application (US20010036026) that there is a tradeoff between the the phase margin loss and the strength of the low frequency vibration suppression. Given the required cutoff frequency of the low pass filter, also known as the Q-filter, it turns out that the relative degree of the Q-filter is the major tuning knob for this tradeoff. The solution in US20010036026 was based on an integer order Q-filter with a variable relative degree. This actually motivated the use of the fractional order Q-filter. The fractional order disturbance observer is based on the fractional order Q-filter. The implementation issue is also discussed. The nice point of this paper is that the traditional DOB is extended to the fractional order DOB with the advantage that the FO-DOB design is now no longer conservative nor aggressive, i.e., given the cutoff frequency and the desired phase margin, we can uniquely determine the fractional order of the low pass filter.

Keywords: Disturbance observer, fractional order calculus, variable relative degree, Q-filter, vibrational suppression, rational approximation, frequency domain fitting.

1. INTRODUCTION

In practice, a physical motion control system will not be the exactly same as a mathematical model no matter how the model is obtained. The disturbance observer regards the difference between the actual output and the output of the nominal model as an equivalent disturbance applied to the nominal model. It estimates the equivalent disturbance and the estimate is utilized as a compensation signal. The

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* For submission to the Special Issue on Fractional Derivatives and Their Applications (FDTA), *Nonlinear Dynamic*. Guest Editors: Professors Om Agrawal, J. A. Tenreiro Machado and Jocelyn Sabatier. October, 2003.

† Corresponding author. Center for Self-Organizing and Intelligent Systems (CSOIS), UMC 4160, College of Engineering, Utah State University, Logan, Utah 84322-4160, USA. Tel. 1(435)797-0148; Fax: 1(435)797-3054. URL: http://www.csois.usu.edu/

disturbance observer (DOB) concept was proposed in [12]. [24] refined the framework of disturbance observer theory based on the design of TDOF (two-degree-of-freedom) servo controllers and the factorization approach. Based on an extended pole placement method and a disturbance observer, an accurate motion controller design was proposed in [2]. Recently, DOB was combined with the zero-phase error tracking algorithm (ZPETC) [23] as reported in [5, 7] for digital implementations. It is now a common practice to use DOB in many high precision motion control systems, e.g., disk drive servo control [3].

Disturbance observers have several attractive features. In the absence of large modelling errors, DOB's allow independent tuning of disturbance rejection characteristics and the command following characteristics. Furthermore, compared to integral action, disturbances observers allow more flexibility via the selection of the order, relative degree, and bandwidth of low-pass filtering known as the disturbance observers filter or Q-filter. It is well known that by appending disturbance states to a traditional state estimator [6], the disturbance compensation can be handled. However, using the disturbance observer structure allows simple and intuitive tuning of the disturbance observer loop gains independent of the state feedback gains. This explains why DOB is more welcome by the control practitioners.

It has been discovered in a recently published US patent application (US20010036026) [3] that the there is a tradeoff between the the phase margin loss and the strength of the low frequency vibration suppression when applying DOB. Given the required cutoff frequency of the Q-filter, it turns out that the relative degree of the Q-filter is the major tuning knob for this tradeoff. As a motivation for the fractional order Q-filter, a solution based on integer order Q-filter with variable relative degree is introduced which is the key contribution of US20010036026 [3]. In this paper, a fractional order disturbance observer based on the fractional order Q-filter is proposed. The fractional order filter is based on the so-called "fractional calculus", a generalization of integration and differentiation to non-integer order operators [13, 22, 11, 19]. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest [8, 20, 16, 17, 21]. For pioneering works, we cite [9, 10, 15, 1]. For the latest development of fractional calculus in automatic control, robotics and signal processing, we cite [25, 4, 14]. The implementation issue is also discussed with an emphasis on the stable minimum phase fitting. The nice point of this paper is that the traditional DOB is extended to fractional order DOB with the advantage that the FO-DOB design is now no longer conservative nor aggressive, i.e., given the cutoff frequency of the Q-filter and the desired phase margin, we can uniquely determine the fractional order of the low pass filter.

This remainder part of this paper is organized as follows. Section 2 introduces the basic concept of disturbance observer (DOB) mainly for vibration attenuation in motion control applications with an extended argument on DOB as a special case of loop-shaping. From practical application point of view, Sec. 3 presents a detailed discussion on the design parameters in DOB and their qualitative effects on system performance. Section 4 highlights the phase margin loss problem associated with the use of DOB. The proposed variable relative degree Q-filter in [3] is briefly introduced in Sec. 5. Then, the proposed fractional order disturbance observer (FO-DOB) is naturally introduced in Sec. 6 with the practical implementation issues discussed in detail in Sec. 7. Finally, concluding remarks are given in Sec. 8.

2. DISTURBANCE OBSERVER (DOB)

In the conventional disturbance observer [12], the basic idea is to use a nominal inverse model of the plant to estimate the disturbance. This is illustrated in Figure 1 where P_n^{-1} is the inverse of the nominal plant model and Q is usually a low pass filter to restrict the effective bandwidth of the DOB. We remark that this DOB configuration is nothing but another form of loopshaping to add more attenuation in the lower frequency range at the cost of the reduced phase margin and the possible amplification of disturbances at other medium and high frequency bands due to the waterbed effect in the sensitivity function. Therefore, it is implicitly implied in DOB that the spectrum of the disturbance d has more low frequency contents than the high frequency ones. We argue that, if d is a white noise, little benefit can be gained from using DOB or any other advanced control technique. Note that if the plant has a nonminimum phase zero, P_n^{-1} will have an unstable mode. So, in this case, the DOB shown in Figure 1, although physically simple, cannot be directly applied. Some modifications will be necessary. In a more general setting, we can redraw the DOB shown in Figure 1 in a form shown in Figure 2 where W is a shaping filter. Note that here the disturbance in front of the plant is considered.

3. ACTUAL DESIGN PARAMETERS IN DOB AND THEIR EFFECTS

In practice, the DOB is usually implemented digitally as shown in Figure 3. In Figure 3, d' is the "observed" disturbance d; $P_n^{-1}(z^{-1})$

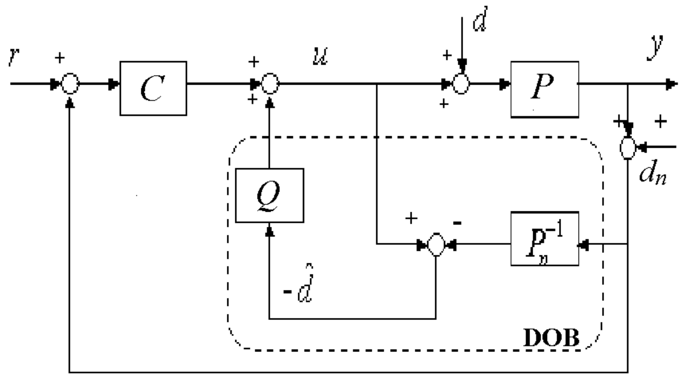

Figure 1. Disturbance Observer Block-diagram - The Conventional Form

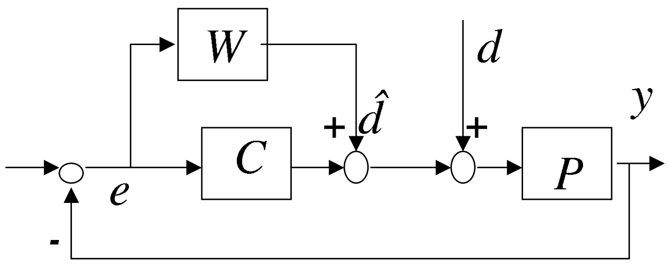

Figure 2. Disturbance Observer Block-diagram - The General Form

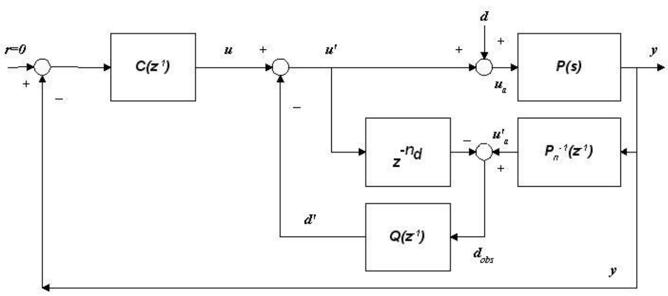

Figure 3. Disturbance Observer Block-diagram - The Digital Form

is the stable inverse of P_n, the nominal model of the actual plant P; n_d is the number of pure delays of the control signal u', the compensated signal of the controller signal u generated by the controller C; Q is a low pass filter with the relative degree n_Q and the cutoff frequency ω_Q.

There are three key parameters in DOB design as shown in Figure 3, namely,

- n_d: the number of pure delays of the control signal u';

- n_Q: the relative degree of Q-filter and
- ω_Q: the cutoff frequency of Q-filter.

In order to see how the overall system based on the disturbance observer behaves, we examine the error transfer function (ETF) $S(j\omega)$ and the disturbance response transfer function (from d to y) $G_{dy}(j\omega)$ from Figure 3. With no DOB,

$$S(j\omega) = \frac{1}{1+PC}, \quad G_{dy}(j\omega) = \frac{P}{1+PC} \tag{1}$$

and with DOB,

$$S(j\omega) = \frac{1}{1+PC+\delta_{PC}}, \quad G_{dy}(j\omega) = \frac{P}{1+PC+\delta_{PC}} \tag{2}$$

where

$$\delta_{PC} = \frac{PP_n^{-1}Q + z^{-n_d}QPC}{1-z^{-n_d}Q} = P\frac{z^{-n_d}Q}{1-z^{-n_d}Q}(P_n^{-1}z^{n_d} + C). \tag{3}$$

Clearly, the disturbance observer cannot be implemented if $Q = 1$. Notice that as in many motion control systems, the nominal plant is in the form of $\frac{K}{(\tau s+1)s}$ and P_n^{-1} in this case can be approximated by an FIR (finite impulse response) filter which is always realizable by itself. Therefore, in this case, no constraint is to be put to the relative degree of Q. In contrary, in the literature, QP_n^{-1} have to be made realizable by letting the relative degree of Q be equal to or greater than that of P_n.

To determine the correct n_d, the major consideration is to minimize the mismatch between the phases of $z^{-n_d}u'$ and u_a as shown in Figure 3. It is found that $n_d = 3$ is the best choice for a high TPI (tracks per inch) disk drive servo system as illustrated by Figure 4 [3]. We comment that, for different applications, n_d should be different and n_d should also include the delay effect of the plant P as also pointed out in [7].

With different relative degree of the Q-filter (n_Q=1,2,3,4), the disturbance attenuation performance is achieved differently. For the lowest relative degree (n_Q =1, note again, as pointed out earlier, n_Q cannot be 0), the best disturbance attenuation is achieved. However, this is at the cost of largest amplification of mid-band frequency contents of both the measurement noises as well as the shock disturbance if any. Therefore, when the disturbance is not presented or is small, n_Q =1 is not a preferred choice. Motivated by this observation, the performance variation with respect to n_Q is illustrated in Figure 5,

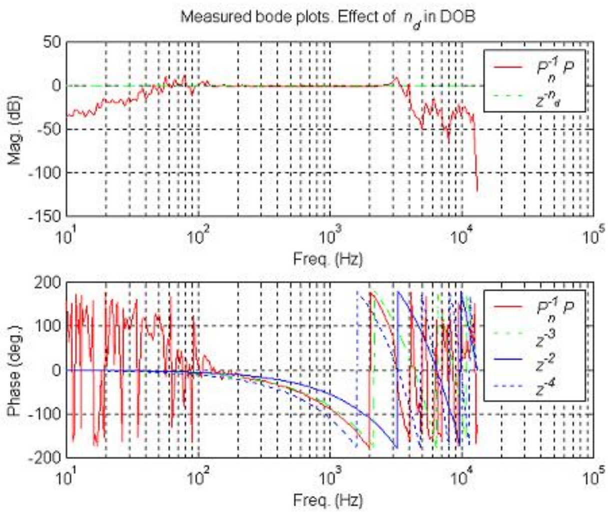

Figure 4. Illustration of the effect of n_d in DOB

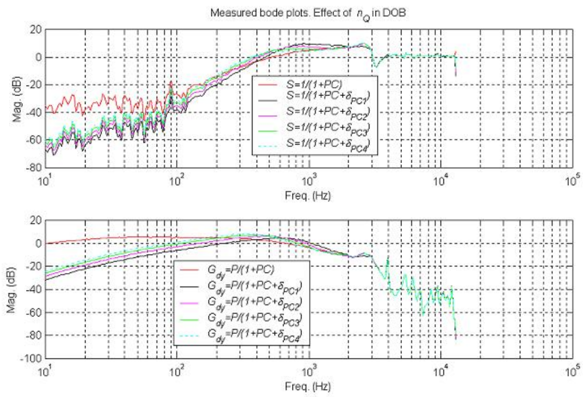

Figure 5. Illustration of the effect of n_Q in DOB

4. LOSS OF THE PHASE MARGIN WITH DOB

The cut off frequency of the Q-filter ω_Q is another key design parameter. Too high a ω_Q may result in worse robustness of the overall system. This can be seen from the variation of PM (phase margin) of the overall closed-loop system. The PM is in fact a function of ω_Q as well as n_Q. The basic trend is that the higher the ω_Q the more PM losses; the larger the n_Q the less PM losses for a fixed ω_Q. This is demonstrated by a set of measured data shown in Figure 6. The red-grid plane represents the PM of the original system. Therefore, Figure 6 can guide us to choose a right combination of ω_Q and n_Q.

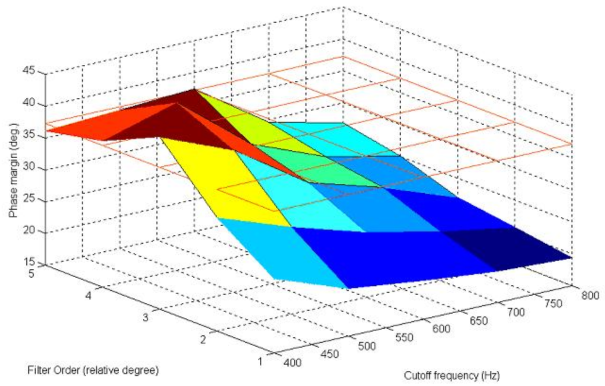

Figure 6. Illustration of phase margin (PM) as a function of n_Q and ω_Q in DOB

It seems quite hard to achieve a good disturbance attenuation performance without loss of PM. A compromise must be made between the disturbance attenuation performance and the robustness of the original system.

5. SOLUTION ONE - RULE BASED SWITCHED LOW PASS FILTERING WITH VARYING RELATIVE DEGREE

This solution is fully documented in [3]. The key motivation is from Figure 6 - the issue of the loss of GM with DOB. In practice, ω_Q should be pre-determined based on the disturbance attenuation requirement. The only trade off tuning knob will be n_Q. A variable relative degree strategy was used [3] where a switching method is applied based on the amplitude of the output y as illustrated in Figure 7. A switching policy used in [3] is shown in Figure 8 for illustration purpose. As a side remark, to avoid the Q-filter initialization problem and the possible big discontinuity in the internal state of Q-filter, all stages of sub-Q-filter in Figure 7 should be run at all times. The explanation of Figure 8 is straightforward and the deadzone is not always required for all DOB applications.

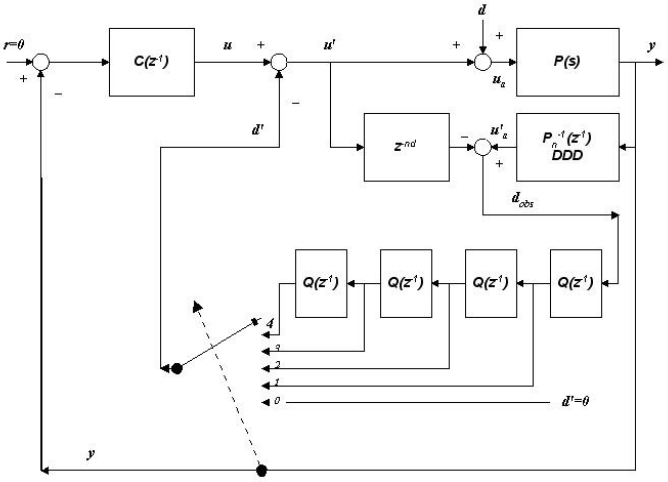

Figure 7. Q-filter in DOB with a varying relative degree

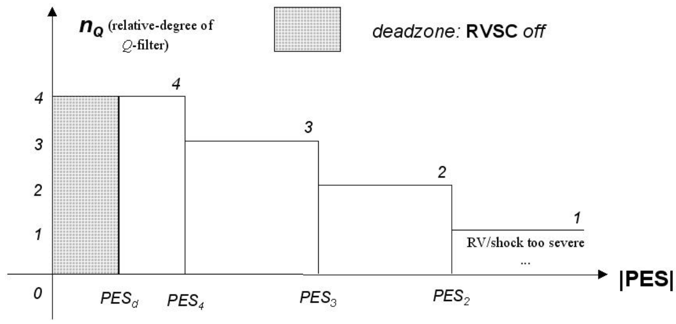

Figure 8. A switching policy for the relative degrees of the Q-filter in DOB

6. THE PROPOSED SOLUTION: GUARANTEED PHASE MARGIN METHOD USING FRACTIONAL ORDER LOW PASS FILTERING

Let us review Figure 6 again. In practice, ω_Q is usually specified by the disturbance attenuation requirement. Moreover, the phase margin of the overall compensated system with DOB is also specified. By Figure 6, we may find that the required n_Q usually lies between two adjacent integers. For example, from the DOB design, it may turn out that Q-filter should be of the following form

$$Q(s) = \frac{1}{(\tau s + 1)^{n_Q}}, \quad n_Q = 3.25 \qquad (4)$$

which is a fractional order low pass filter (FO-LPF). When we use a fractional order Q-filter in DOB, we call it "fractional order disturbance observer".

Clearly, the fractional order filter introduced above is based on the concept of fractional calculus, a generalization of integration and differentiation to non-integer order operators [13, 22, 11, 19]. There are two commonly- used definitions for general fractional differentiation and integral, i.e., the Grünwald definition and the Riemann-Liouville definition [13, 11, 19]. The Grünwald definition is that

$$ {}_a\mathrm{D}_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{5} $$

where $[\cdot]$ is a flooring-operator, while the Riemann-Liouville definition is given by

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \tag{6} $$

for $(n-1 < \alpha < n)$ and where $\Gamma(x)$ is the well known Euler's `gamma` function. The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the Riemann-Liouville fractional derivative (6) has the form [18]:

$$ \int_0^\infty e^{-pt} \, {}_0\mathrm{D}_t^\alpha f(t) \, \mathrm{d}t = $$
$$ = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \; {}_0\mathrm{D}_t^{\alpha-k-1} f(t) \Big|_{t=0}, \tag{7} $$

for $(n-1 < \alpha \leq n)$.

7. IMPLEMENTATION ISSUES: STABLE MINIMUM-PHASE FREQUENCY DOMAIN FITTING

From the definitions of fractional-order derivative, we know that any controller involving a fractional order differentiator or integrator is in fact a filter with infinite (integer) order, or, we can say that the filter is with infinite length of memory. In implementing a given FOC, its frequency response is actually exactly known. Clearly, for a given range of frequency of interest, say, $\omega \in [\omega_L, \omega_H]$, a set of frequency response data (magnitude and phase) can be obtained. We can take this set of data as the *measured frequency response* data set and feed it to any frequency-domain system identification software package.

At this point, one may think of the ready Matlab function `invfreqs` or `invfreqz` in the Matlab Signal Processing Toolbox. But this does *not* work for our purpose here mainly due to the bad numerical conditioning in the algorithms used in `invfreqs` or `invfreqz`. It is found that the `ELiS` function in the Matlab Frequency Domain Identification Toolbox works fine for our transfer function fitting here. In particular, the stability of the fit transfer function can be guaranteed. Another attractive feature is its professionally designed GUI [1].

A simple command line example is given by the following script:

```
%Istvan Kollar's stable transfer function fitting
f=logspace(-1,3,200)'; % freq. band of practical interest
Y=1./(j*2*pi*f).^(1/2);% the desired freq. response 1/s^.5
U=ones(size(Y));    % set input to 1 to get I/O data
d=fiddata(Y,U,f);   % build the FIDdata
d.variance=[0,1e-6]; % artificial variance
if ~exist('order'), order=9; end order=yesinput('Order of
model',order,[1,inf]);
%First search for best cost function with stabilization:
disp('First search for best fit among trials...')
[m,finf]=elis(d,'s',order,order,...
    struct('stabilization','r','forceminimumphase','r',...
    'plotdens',-inf,'plot0','off'),...
    struct('displaymessages','off'));
[cfm,itmax]=min(finf.cfv); itmax=itmax-1;
fprintf('Iterate until best fit, itmax=%.0f ...\n',itmax)
figure(1), clf
iterctrl % allow manual iter. on fig. (select 'Finish')
% return model object 'm', order/order, forcing stability
m=elis(d,'s',order,order,...
struct('stabilization','r','forceminimumphase','r',...
'itmax',itmax)); figure(2), plotelpz(m),
zoom on % plot the pole-zero distribution
if ~exist('bi'), bi=2; end,
if order==4, bi=2; end, bi=bi+1;
figure(bi), plot(m), zoom on  % plot the Bode magnitude
xlabel(sprintf('Order: %.0f/%.0f',order,order));drawnow
```

Using `get(m)`, we have

```
Version = 2.2
Date = '02-Dec-2000 15:55:26'
```

[1] Refer to `FDIDENT` home page `http://elecwww.vub.ac.be/fdident`.

```
Data = [2x1x200 fiddata]
Algorithm = [1x1 struct]
Variable = 's'
Representation = 'polynomial'
num = [-6.58e-36 -4.56e-31 -4.34e-27 -1.13e-23 -9.4e-21
    -2.46e-18 -1.88e-16 -3.61e-15 -1.44e-14 -8.7537e-15]
denom = [-2.46e-33 -5.33e-29 -2.57e-25 -3.76e-22 -1.76e-19
    -2.53e-17 -9.88e-16 -8.9e-15 -1.48e-14 -2.49e-15]
FreqVect = [200x1 double]
Fscale = 1
Delay = 0
Covariance = [21x21 double]
FitInfo = [1x1 struct]
```

Entering the above fit transfer function coefficients into `CtrlLAB`, the most downloaded package developed by Professor Dingyü Xue for SISO (single input single output) control system analysis and design in Matlab Central[2], we can get, via several mouse clicks, the Bode plot and Nichols chart shown in Figure 9 and Figure 10 respectively.

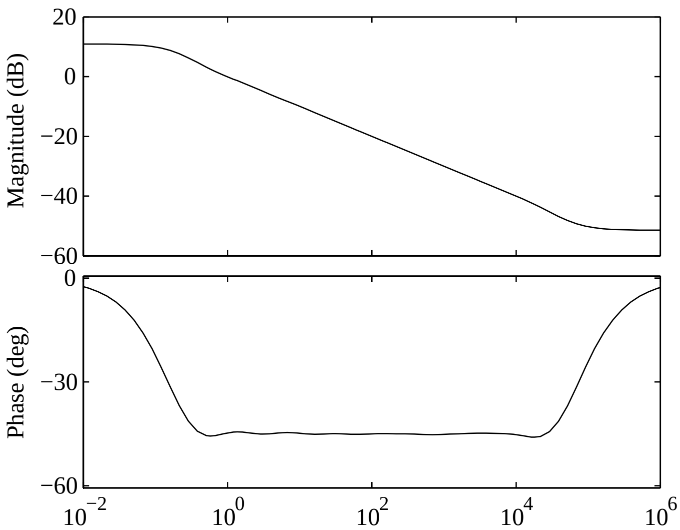

Figure 9. Bode plot for order 9/9 - Stable frequency domain fitting of $1/\sqrt{s}$

Making a good fit with stable poles is sometimes rather difficult. In a high order fitting, there is a good chance that some poles will be driven to the unstable region due to numerical sensitivity prob-

[2] Matlab Central URL: `http://www.mathworks.com/Matlabcentral/fileexchange/loadFile.do?objectId=18&objectType=file`

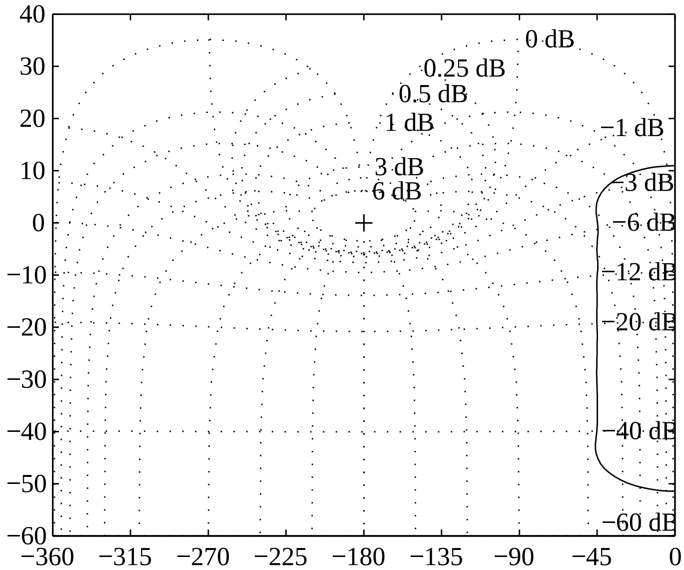

Figure 10. Nichols chart for order 9/9 - Stable frequency domain fitting of $1/\sqrt{s}$

lems. The Matlab Frequency Domain System Identification Toolbox (Version 3.0 of 22-Nov-00) offers some (artificial) tools to force stable solutions. In the stable fitting script here, the fit transfer function is also restricted to be minimum phase. This is achieved by requesting the reflection/contraction of the unstable zeros and poles.

As a benchmark fitting result, consider the following general filter in lead/lag form [21]

$$C_r(s) = C_0 \left(\frac{1 + s/\omega_b}{1 + s/\omega_h} \right)^r \tag{8}$$

where $0 < \omega_b < \omega_h$, $C_0 > 0$ and $r \in (0, 1)$. Here we give out the fitting result for $C_{0.65}(s)$ [21] with $C_0 = 4280.1$, $\omega_b = 0.5$, $\omega_h = 200$ using the stable frequency fitting method introduced in this section. The 4/4 fitting result is that

$$\begin{aligned} C_{0.65}(s) &= 4280.1 \left(\frac{1+2s}{1+0.005s} \right)^{0.65} \\ &\approx \frac{9.457\times10^{-11}s^4+1.218\times10^{-8}s^3+3.07\times10^{-7}s^2+1.476\times10^{-6}s+9.794\times10^{-7}}{4.5\times10^{-16}s^4+1.161\times10^{-13}s^3+6.99\times10^{-12}s^2+9.516\times10^{-11}s+2.14\times10^{-10}} \end{aligned}$$

with its Bode plot and Nichols chart drawn via CtrlLAB in Figure 11 and Figure 12, respectively. We can see that Figure 11 is quite similar to the characteristic of a frequency-band fractional differentiator.

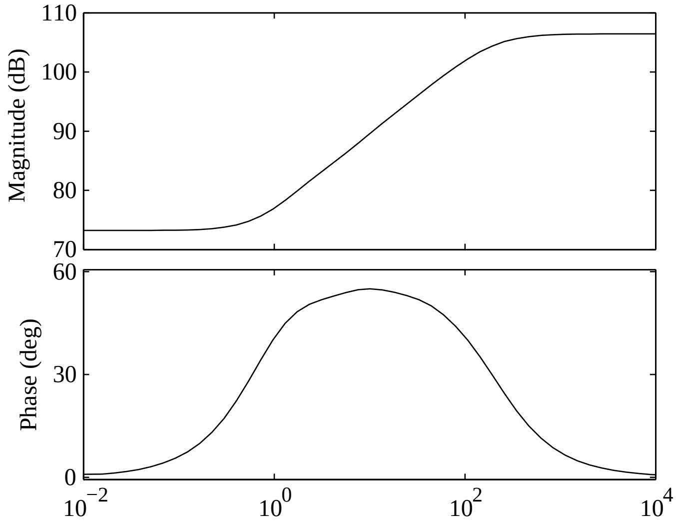

Figure 11. Bode plot for order 4/4 - Stable frequency domain fitting of $C_{0.65}(s)$

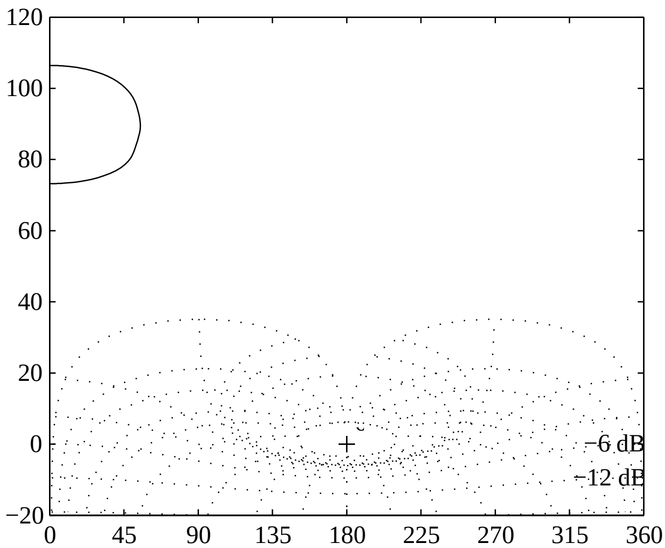

Figure 12. Nichols chart for order 4/4 - Stable frequency domain fitting of $C_{0.65}(s)$

8. CONCLUDING REMARKS

We proposed to use the fractional order disturbance observer (FO-DOB) for vibration suppression applications such as hard disk drive servo control. The motivation is explained in detail. The major problem is the tradeoff between the the phase margin loss and the strength of the low frequency vibration suppression. Given the required cutoff frequency of the Q-filter, it turns out that the relative degree of the Q-filter is the major tuning knob for this tradeoff. To motivate the introduction of the fractional order Q-filter, an existing solution based on integer order Q-filter with a variable relative degree is introduced which is the key contribution of US20010036026 [3]. The fractional order disturbance observer based on the fractional order Q-filter is proposed with the implementation method discussed. The nice point of this paper is that the traditional DOB has been extended to the fractional order DOB with the advantage that the FO-DOB design is now no longer conservative or aggressive, i.e., given the cutoff frequency and the desired phase margin, we can uniquely determine the fractional order of the low pass filter.

We comment that the experimentation verification of the proposed fractional order disturbance observer (FO-DOB) seems to be interesting but straightforward in view of the explanation in this paper and the existing implemented results in [3]. However, during the experimental verification of FO-DOB, there might be some new issues emerging such as the nonlinearity effects.

Acknowledgment

The first author would like to thank his ex-colleagues and co-inventors, Ooi, Kian Keong, Ding, Ming Zhong, Tan, Lee Ling and Soh, Kok Tong in Singapore Science Park Design Center, Seagate Technology.

This project has been funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program/Twinning Program supported by Contract No. INT-0002341 from the National Science Foundation. The contents of this publication do not necessarily reflect the views or policies of the National Academy of Sciences or the National Science Foundation, nor does mention of trade names, commercial products or organizations imply endorsement by the National Academy of Sciences or the National Science Foundation.

Blas M. Vinagre is partially supported by the Research Grant (Junta de Extremadura and FEDER) 2PR02A024.

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