

Using Continued Fraction Expansion to Discretize Fractional Order Derivatives *

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Abstract. This paper introduces the continued fraction expansion (CFE) method to directly discretize the fractional order differentiators defined in continuous time domain. Unlike the use of conventional generating functions such as Euler operator, Tustin operator etc., a second order IIR (infinite impulse response) type generating function is introduced, which is obtained by the stable inversion of the weighted sum of Simpson integration formula and the trapezoidal integration formula. The distinguishing point of the proposed discretization method lies in an additional tuning knob to compromise the high frequency approximation accuracy. It is also shown that the Tustin operator based discretization is a special case of the proposed scheme. Numerical examples are included for illustrations.

Keywords: Fractional differentiator, fractional-order dynamic systems, fractional-order differentiator, discretization, Tustin operator, Al-Alaoui operator.

1. INTRODUCTION

Although the fractional order calculus is a 300-years-old topic, the theory of fractional-order derivative was developed mainly in the 19-th century. Recent books [15, 28, 14, 25] provide a good source of references on fractional calculus. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest [10, 26, 21, 22, 27]. For pioneering works, we cite [12, 13, 17, 4]. For recent developments, we cite [35, 8, 16]. In most cases, our objective is to apply fractional order control to enhance the system control

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performance. For example, as in the CRONE¹ control [18, 21, 22], *fractal robustness* is pursued. The desired frequency template leads to fractional transmittance [23, 20] on which the CRONE controller synthesis is based. In the CRONE controller, the major ingredient is the fractional-order derivative s^r , where r is a real number and s is the Laplacian operator. Another example is the $PI^\lambda D^\mu$ controller [26, 24], an extension of PID controller. In general form, the transfer function of $PI^\lambda D^\mu$ is given by $K_p + T_i s^{-\lambda} + T_d s^\mu$, where λ and μ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $\lambda = 0$ ($T_i = 0$) we obtain a PD^μ controller, etc. All these types of controllers are particular cases of the $PI^\lambda D^\mu$ controller. It can be expected that the $PI^\lambda D^\mu$ controller may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, $PI^\lambda D^\mu$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of frequencies of practical interest [19, 20]. Moreover, the fractional order can be a complex number as discussed in [19]. In this paper, we focus on the case where the fractional order is a real number.

The key step in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r . In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods [19], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s -transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator [11, 31–33], continuous fractional expansion (CFE) of the Tustin operator [31–33, 6, 34], and numerical integration based method [11, 6, 5]. However, as pointed out in [1–3], the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [6] which yields the Al-Alaoui operator [1].

Recently, there are some methods to directly obtain the digital fractional order differentiators (DFOD) in FIR (finite impulse response) form [30, 29]. However, using an FIR filter to approximate s^r may

¹ CRONE is a French abbreviation for “*Contrôle Robuste d’Ordre Non Entier*” (which means non-integer order robust control).

be less efficient due to the very high order of the FIR filter. In this paper, based on [7], a new IIR (infinite impulse response) type digital fractional order differentiator is proposed by using a new family of first order digital differentiators expressed in a second order IIR filter form. The integer first order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule [2]. The distinguishing point of the proposed scheme lies in an additional tuning knob to compromise the high frequency approximation accuracy with the Tustin operator based discretization as the special case.

This paper is organized as follows: in Sec. 2, the fractional order derivative and its discretization are briefly introduced. Section 3 introduces the basic concept of generating function when performing the discretization. The IIR-type integer first order digital integrator by weighted Simpson and Tustin schemes is presented in Sec. 4. Section 5 presents the IIR-type DFOD based on the generating function obtained via stable inversion of the integrator presented in Sec. 4. Some illustrative examples are presented in Sec. 6. Section 7 is for the special case of using Tustin operator alone. Finally, Sec. 8 concludes this paper.

2. FRACTIONAL-ORDER DERIVATIVE AND ITS DISCRETIZATION

Fractional calculus generalizes the integration and differentiation to the non-integer (fractional) order fundamental operator ${}_aD_t^r$, where a and t are the limits and r , ($r \in \mathbb{R}$) the order of the operation. The two definitions used for the general fractional integrodifferential are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [15], [25]. The GL definition is that

$${}_aD_t^r f(t) = \lim_{h \rightarrow 0} h^{-r} \sum_{j=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^j \binom{r}{j} f(t - jh) \quad (1)$$

where $\lfloor \cdot \rfloor$ means the integer part while the RL definition

$${}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} d\tau \quad (2)$$

for $(n-1 < r < n)$ and where $\Gamma(\cdot)$ is the Euler's *gamma* function.

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation[25]. The Laplace transform of the RL fractional derivative/integral (2) under zero initial conditions

for order r , ($0 < r < 1$) is given by [15]:

$$\mathcal{L}\{ {}_a D_t^{\pm r} f(t); s \} = s^{\pm r} F(s) \quad (3)$$

The key point in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods, two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s -transfer function. In this paper, we focus on the *direct discretization* method.

The simplest and most straightforward method is the direct discretization using finite memory length expansion from the GL definition (1). This approach is based on the fact that, for a wide class of functions, the two definitions - GL (1) and RL (2) - are equivalent [25]. In general, the discretization of fractional-order differentiator/integrator $s^{\pm r}$, ($r \in \mathbb{R}$) can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [9]. For example, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$, performing the PSE of $(1 - z^{-1})^{\pm r}$ gives the discretization formula for GL formula (1). By using the short memory principle [25], the discrete equivalent of the fractional-order integro-differential operator, $(\omega(z^{-1}))^{\pm r}$, is given by

$$(\omega(z^{-1}))^{\pm r} = T^{\mp r} z^{-[L/T]} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j} \quad (4)$$

where T is the sampling period, L is the memory length and $(-1)^j \binom{\pm r}{j}$ are binomial coefficients $c_j^{(r)}$, ($j = 0, 1, \dots$) where

$$c_0^{(r)} = 1, \quad c_j^{(r)} = \left(1 - \frac{1 + (\pm r)}{j} \right) c_{j-1}^{(r)}. \quad (5)$$

It is very important to note that PSE scheme leads to approximations in the form of polynomials, that is, the discretized fractional order derivative is in the form of FIR filters. Taking into account that our aim is to obtain discrete equivalents to the fractional integrodifferential operators in the Laplace domain, $s^{\pm r}$, the following considerations have to be made:

1. s^r , ($0 < r < 1$), viewed as an operator, has a branch cut along the negative real axis for arguments of s on $(-\pi, \pi)$ but is free of poles and zeros.

2. A dense interlacing of simple poles and zeros along a line in the s plane is, in some way, equivalent to a branch cut.
3. It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, rational approximations frequently converge much more rapidly than PSE and have a wider domain of convergence in the complex plane.
4. Trapezoidal rule maps adequately the stability regions of the s plane on the z plane, and maps the points $s = 0, s = -\infty$ to the points $z = 1$ and $z = -1$ respectively.

3. THE CONCEPT OF GENERATING FUNCTION

In general, the discretization of the fractional-order differentiator s^r (r is a real number) can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [9]. For example, as shown in the last section, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$ with T the sampling period, performing the power series expansion (PSE) of $(1 - z^{-1})^{\pm r}$ gives the discretization formula which is actually in FIR filter form [11, 31]. In [33, 6], the trapezoidal (Tustin) rule is used as a generating function

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^{\pm r}. \quad (6)$$

The DFOD is then obtained by using the CFE [33] or a new recursive expansion formula [6]. It is interesting to note that in [6], the so-called Al-Alaoui operator is used which is a mixed scheme of Euler and Tustin operators [1]. Correspondingly, the generating function for discretization is

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{8}{7T} \frac{1 - z^{-1}}{1 + z^{-1}/7} \right)^{\pm r}. \quad (7)$$

Clearly, (7) is an infinite order of rational discrete-time transfer function. To approximate it with a finite order rational one, continued fraction expansion (CFE) is an efficient way. In general, any function

$G(z)$ can be represented by continued fractions in the form of

$$G(z) \simeq a_0(z) + \frac{b_1(z)}{a_1(z) + \frac{b_2(z)}{a_2(z) + \frac{b_3(z)}{a_3(z) + \dots}}} \quad (8)$$

where the coefficients a_i and b_i are either rational functions of the variable z or constants. By truncation, an approximate rational function, $\widehat{G}(z)$, can be obtained.

The basic idea of this paper is to define a new generating function by applying a new family of first order digital differentiators expressed in the second order IIR filter form [2] via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

4. INTEGER ORDER IIR TYPE DIGITAL INTEGRATOR BY WEIGHTED SIMPSON AND TUSTIN SCHEMES

It was pointed out in [1, 3] that the magnitude of the frequency response of the ideal integrator $1/s$ lies between that of the Simpson and trapezoidal digital integrators. It is reasonable to “interpolate” the Simpson and trapezoidal digital integrators to compromise the high frequency accuracy in frequency response. This leads to the following hybrid digital integrator

$$H(z) = aH_S(z) + (1 - a)H_T(z), \quad a \in [0, 1] \quad (9)$$

where a is actually a weighting factor or tuning knob. $H_S(z)$ and $H_T(z)$ are the z -transfer functions of the Simpson’s and the trapezoidal integrators given respectively as follows:

$$H_S(z) = \frac{T(z^2 + 4z + 1)}{3(z^2 - 1)} \quad (10)$$

and

$$H_T(z) = \frac{T(z + 1)}{2(z - 1)}. \quad (11)$$

The overall weighted digital integrator with the tuning parameter a is hence given by

$$\begin{aligned} H(z) &= \frac{T(3 - a)\{z^2 + [2(3 + a)/(3 - a)]z + 1\}}{6(z^2 - 1)} \\ &= \frac{T(3 - a)(z + r_1)(z + r_2)}{6(z^2 - 1)} \end{aligned} \quad (12)$$

where

$$r_1 = \frac{3 + a + 2\sqrt{3a}}{3 - a}, \quad r_2 = \frac{3 + a - 2\sqrt{3a}}{3 - a}.$$

It is interesting to note the fact that $r_1 = \frac{1}{r_2}$ and $r_1 = r_2 = 1$ only when $a = 0$ (trapezoidal). For $a \neq 0$, $H(z)$ must have one non-minimum phase (NMP) zero.

5. IIR TYPE FRACTIONAL ORDER DIGITAL DIFFERENTIATOR

Firstly, we can obtain a family of new integer order digital differentiators from the digital integrators introduced in the last section. Direct inversion of $H(z)$ will give an unstable filter since $H(z)$ has an NMP zero r_1 . By reflecting the NMP r_1 to $1/r_1$, i.e. r_2 , we have

$$\tilde{H}(z) = K \frac{T(3-a)(z+r_2)^2}{6(z^2-1)}.$$

To determine K , let the final value of the impulse responses of $H(z)$ and $\tilde{H}(z)$ be the same, i.e., $\lim_{z \rightarrow 1} (z-1)H(z) = \lim_{z \rightarrow 1} (z-1)\tilde{H}(z)$, which gives $K = r_1$. Therefore, the new family of the digital differentiators are given by

$$\omega(z) = \frac{1}{\tilde{H}(z)} = \frac{6(z^2-1)}{r_1 T(3-a)(z+r_2)^2} = \frac{6r_2(z^2-1)}{T(3-a)(z+r_2)^2}. \quad (13)$$

We can regard $\omega(z)$ in (13) as the generating function introduced in the last section. Finally, we can obtain the expression for the DFOD as

$$G(z^{-1}) = \left(\omega(z^{-1})\right)^r = k_0 \left(\frac{1-z^{-2}}{(1+bz^{-1})^2}\right)^r \quad (14)$$

where $r \in [0, 1]$, $k_0 = \left(\frac{6r_2}{T(3-a)}\right)^r$ and $b = r_2$.

It is well known that, compared to the power series expansion method, the Continued Fraction Expansion (CFE) is a method for evaluation of functions with faster convergence and larger domain of convergence in the complex plane. Using CFE, an approximation for a irrational function $G(z^{-1})$ can be expressed in the form of (8). Similar to (7), here, the irrational transfer function $G(z^{-1})$ in (14) can be expressed by an infinite order of rational discrete-time transfer function by CFE method as shown in (8).

The CFE expansion can be automated by using a symbolic computation tool such as the MATLAB Symbolic Toolbox. To illustrate,

let us denote $x = z^{-1}$. Referring to (14), the task is to perform the following expansion:

$$\text{CFE} \left(\frac{1 - x^2}{(1 + bx)^2} \right)^r$$

to the desired order n . The following MATLAB script will generate the above CFE with `p1` and `q1` containing, respectively, the numerator and denominator polynomials in x or z^{-1} with their coefficients being functions of b and r .

```
clear all;close all;syms x r b;maple('with(numtheory)');
aas = ( (1-x*x)/(1+b*x)^2 )^r; n=3; n2=2*n;
maple(['cfe := cfrac(' char(aas) ',x,n2);']);
pq=maple('P_over_Q := nthconver', 'cfe', n2);
p0=maple('P := nthnumer', 'cfe', n2);
q0=maple('Q := nthdenom', 'cfe', n2);
p=(p0(5:length(p0))); q=(q0(5:length(q0)));
p1=collect(sym(p),x); q1=collect(sym(q),x);
```

6. ILLUSTRATIVE EXAMPLES

Here we present some results for $r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. Let $T = 0.001$ sec. We have the following:

$$\begin{aligned} G_{(2,0.00)}(z^{-1}) &= \frac{178.9 - 89.44z^{-1} - 44.72z^{-2}}{4 + 2z^{-1} - z^{-2}} \\ G_{(2,0.25)}(z^{-1}) &= \frac{138.8 + 98.07z^{-1} - 158.2z^{-2}}{4 + 5.034z^{-1} - z^{-2}} \\ G_{(2,0.50)}(z^{-1}) &= \frac{127 + 41.26z^{-1} - 112.6z^{-2}}{4 + 2.98z^{-1} - z^{-2}} \\ G_{(2,0.75)}(z^{-1}) &= \frac{119.3 + 25.56z^{-1} - 97.96z^{-2}}{4 + 2.19z^{-1} - z^{-2}} \\ G_{(2,1.00)}(z^{-1}) &= \frac{113.4 + 17.74z^{-1} - 89.81z^{-2}}{4 + 1.698z^{-1} - z^{-2}} \end{aligned} \tag{15}$$

$$\begin{aligned} G_{(3,0.00)}(z^{-1}) &= \frac{357.8 - 178.9z^{-1} - 178.9z^{-2} + 44.72z^{-3}}{8 + 4z^{-1} - 4z^{-2} - z^{-3}} \\ G_{(3,0.25)}(z^{-1}) &= \frac{392.9 - 78.04z^{-1} - 349.8z^{-2} + 88.97z^{-3}}{11.32 + 4z^{-1} - 5.66z^{-2} - z^{-3}} \end{aligned}$$

$$\begin{aligned}
G_{(3,0.50)}(z^{-1}) &= \frac{1501 - 503.6z^{-1} - 1289z^{-2} + 446.5z^{-3}}{47.26 + 4z^{-1} - 23.63z^{-2} - z^{-3}} \\
G_{(3,0.75)}(z^{-1}) &= \frac{968.1 - 442z^{-1} - 820.8z^{-2} + 363z^{-3}}{32.47 - 4z^{-1} - 16.24z^{-2} + z^{-3}} \\
G_{(3,1.00)}(z^{-1}) &= \frac{353.1 - 208z^{-1} - 297.4z^{-2} + 164.7z^{-3}}{12.46 - 4z^{-1} - 6.228z^{-2} + z^{-3}}
\end{aligned} \tag{16}$$

$$\begin{aligned}
G_{(4,0.00)}(z^{-1}) &= \frac{715.5 - 357.8z^{-1} - 536.7z^{-2} + 178.9z^{-3} + 44.72z^{-4}}{16 + 8z^{-1} - 12z^{-2} - 4z^{-3} + z^{-4}} \\
G_{(4,0.25)}(z^{-1}) &= \frac{555.3 - 392.9z^{-1} - 477.2z^{-2} + 349.8z^{-3} - 19.56z^{-4}}{16 - 2.489z^{-1} - 12z^{-2} + 1.245z^{-3} + z^{-4}} \\
G_{(4,0.50)}(z^{-1}) &= \frac{508.1 - 1501z^{-1} - 4.478z^{-2} + 1289z^{-3} - 382.9z^{-4}}{16 - 40.54z^{-1} - 12z^{-2} + 20.27z^{-3} + z^{-4}} \\
G_{(4,0.75)}(z^{-1}) &= \frac{477 + 968.1z^{-1} - 919z^{-2} - 820.8z^{-3} + 422.7z^{-4}}{16 + 37.8z^{-1} - 12z^{-2} - 18.9z^{-3} + z^{-4}} \\
G_{(4,1.00)}(z^{-1}) &= \frac{453.6 + 353.1z^{-1} - 661.7z^{-2} - 297.4z^{-3} + 221.5z^{-4}}{16 + 16.74z^{-1} - 12z^{-2} - 8.371z^{-3} + z^{-4}}
\end{aligned} \tag{17}$$

The Bode plot comparisons for the above three groups of approximate fractional order digital differentiators are summarized in Figure 1, Figure 2 and Figure 3 respectively. We can clearly observe the improvement in high frequency magnitude response. If trapezoidal scheme is used, the high frequency magnitude response is far from the ideal one. The role of the tuning knob a is obviously useful in some applications. MATLAB code for this new DFOD is available upon request.

REMARK 6.1. *The phase approximations in Figure 1, Figure 2 and Figure 3 did not consider the linear phase lag compensation as did in [6]. For a given a and r , a pure linear phase lead compensation can be added without affecting the magnitude approximation. For example, when $a = r = 0.5$, a pure phase lead $z^{0.5}$ can be cascaded to $G_{(4,0.50)}(z^{-1})$ and the phase approximation can be improved as shown in Fig. 4. Note that $z^{0.5}$ can be realized by $z^{-0.5}/z^{-1}$ to be causally realizable.*

For $n = 3$ and $n = 4$, the pole-zero maps are shown respectively in Figure 5 and Figure 6 with some different values of a . First of all, we observe that there is no complex conjugate poles or zeros. We can further observe that for odd order of CFE ($n = 3$), the pole-zero maps are nicely behaved, that is, all the poles and zeros lie inside the unit circle and the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$. However, when n is even, and

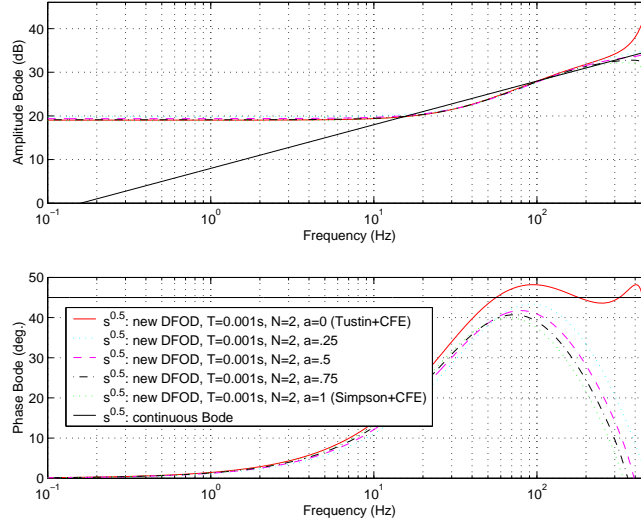


Figure 1. Bode plot comparison for $r = 0.5$, $n = 2$ and $a = 0, .25, .5, .75, 1$

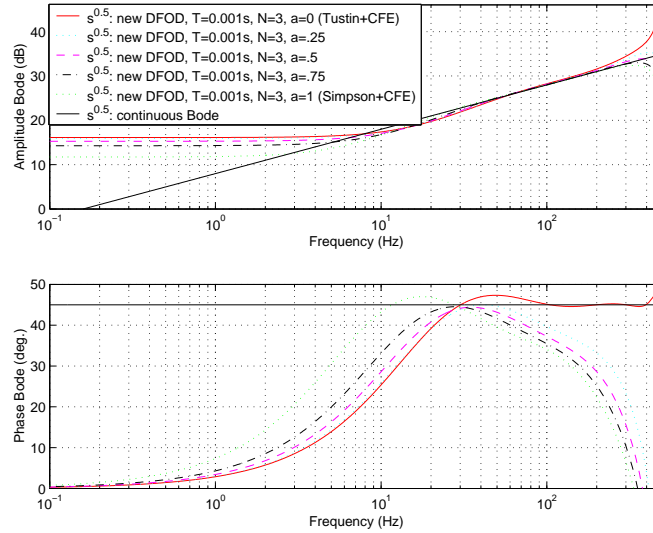


Figure 2. Bode plot comparison for $r = 0.5$, $n = 3$ and $a = 0, .25, .5, .75, 1$

when a is near to 1, there may have one cancelling pole-zero pair as seen in Figure 6 which may not be desirable. We suggest to use an odd n when applying the new discretization scheme of this paper. As can be seen in the next section, when $a = 0$, the pole-zero map is always inside the unit circle in an interlacing way along the segment of the real axis corresponding to $z \in (-1, 1)$.

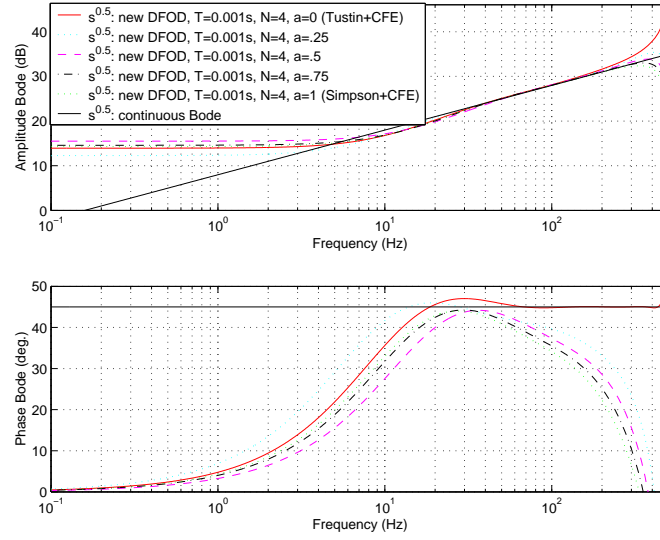


Figure 3. Bode plot comparison for $r = 0.5$, $n = 4$ and $a = 0, .25, .5, .75, 1$

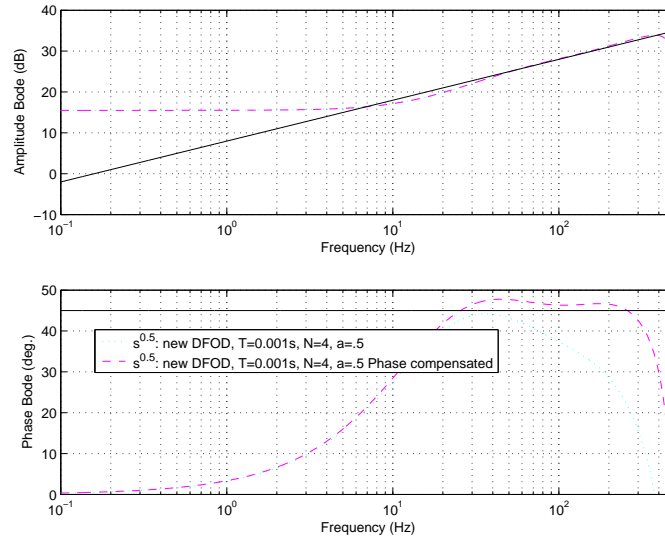


Figure 4. Effect of linear phase compensation for $r = 0.5$, $n = 4$ and $a = .5$

7. THE SPECIAL CASE: TUSTIN OPERATOR ALONE

Consider the special case for $a = 0$ in the general scheme(14). This is the CFE of Tustin operator. Let the resulting discrete transfer function,

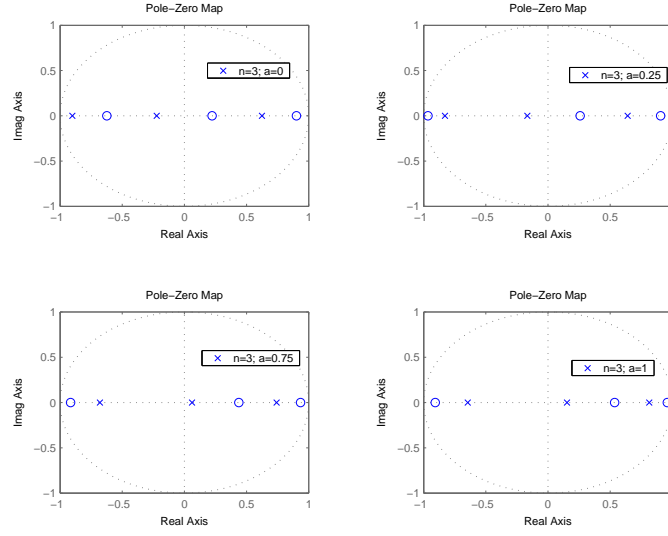


Figure 5. Pole-zero maps for $r = 0.5$, $n = 3$ and $a = 0, .25, .75, 1$

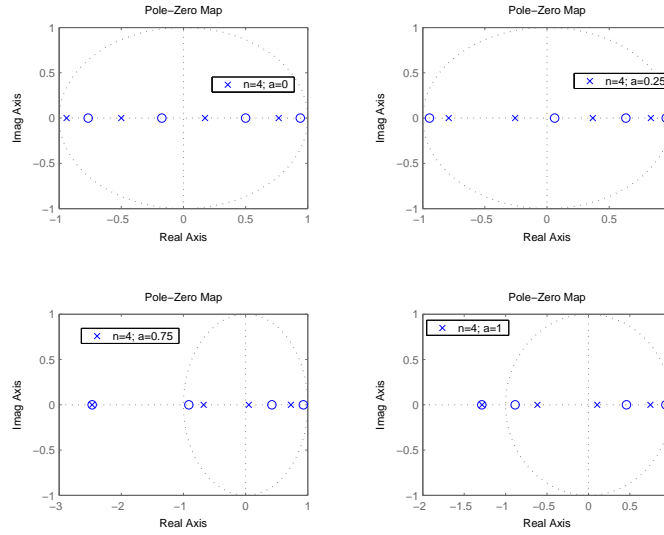


Figure 6. Pole-zero maps for $r = 0.5$, $n = 4$ and $a = 0, .25, .75, 1$

approximating fractional-order operators, be expressed by

$$\begin{aligned}
 D^{\pm r}(z) &= \frac{Y(z)}{F(z)} = \left(\frac{2}{T}\right)^{\pm r} \text{CFE} \left\{ \left(\frac{1-z^{-1}}{1+z^{-1}} \right)^{\pm r} \right\}_{p,q} \\
 &= \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}
 \end{aligned} \tag{18}$$

where T is the sample period, $\text{CFE}\{u\}$ denotes the function resulting from applying the continued fraction expansion to the function u , $Y(z)$ is the Z transform of the output sequence $y(nT)$, $F(z)$ is the Z transform of the input sequence $f(nT)$, p and q are the orders of the approximation, and P and Q are polynomials of degrees p and q , correspondingly, in the variable z^{-1} .

By using the MAPLE call

```
Drp:=cfrac(((1-x)/(1+x))^r,x,p)
```

where $x = z^{-1}$, the obtained symbolic approximation has the following form:

$$D^r(z) = 1 + \frac{z^{-1}}{-\frac{1}{2} \frac{1}{r} + \frac{z^{-1}}{-2 + \frac{3}{2} \frac{r}{r^2 - 1} + \frac{z^{-1}}{2 + \frac{5}{2} \frac{r^2 - 1}{r(-4 + r^2)} + \frac{z^{-1}}{-2 + \dots}}}}. \quad (19)$$

In MATLAB Symbolic Toolbox, we can get the same result by the following script:

```
syms x r;maple('with(numtheory)');
f = ((1-x)/(1+x))^r;
maple(['cf := cfrac(' char(f) ',x,10);'])
maple('nd5 := nthconver','cf',10)
maple('num5 := nthnumer','cf',10)
maple('den5 := nthdenom','cf',10)
```

In Table 7, the general expressions for numerator and denominator of $D^r(z)$ in (18) are listed for $p = q = 1, 3, 5, 7, 9$.

With $r = 0.5$ and $T = 0.001$ sec. the approximate models for $p = q = 1, 3, 7, 9$ are:

$$G_1(z) = 44.72 \frac{z - 0.5}{z + 0.5}, \quad G_3(z) = 44.72 \frac{z^3 - 0.5z^2 - 0.5z + 0.125}{z^3 + 0.5z^2 - 0.5z - 0.125}$$

$$G_7(z) = 44.72 \frac{z^7 - 0.5z^6 - 1.5z^5 + 0.625z^4 + 0.625z^3 - 0.1875z^2 - 0.0625z + 0.007813}{z^7 + 0.5z^6 - 1.5z^5 - 0.625z^4 + 0.625z^3 + 0.1875z^2 - 0.0625z - 0.007813}$$

$$G_9(z) = 44.72 \frac{z^9 - 0.5z^8 - 2z^7 + 0.875z^6 + 1.313z^5 - 0.4688z^4 - 0.3125z^3 + 0.07813z^2 + 0.01953z - 0.001953}{z^9 + 0.5z^8 - 2z^7 - 0.875z^6 + 1.313z^5 + 0.4688z^4 - 0.3125z^3 - 0.07813z^2 + 0.01953z + 0.001953}$$

In Figure 7, the Bode plots and the distributions of zeros and poles of the approximations are presented. In Figure 7a the effectiveness of the

Table I. $D^r(z)$

$\mathbf{p=q}$	$\mathbf{P_p(z^{-1}) (k = 1), \text{ and } Q_q(z^{-1})(k = 0)}$
1	$(-1)^k z^{-1}r + 1$
3	$(-1)^k (r^3 - 4r)z^{-3} + (6r^2 - 9)z^{-2} + (-1)^k 15z^{-1}r + 15$
5	$(-1)^k (r^5 - 20r^3 + 64r)z^{-5} + (-195r^2 + 15r^4 + 225)z^{-4} + (-1)^k (105r^3 - 735r)z^{-3} + (420r^2 - 1050)z^{-2} + (-1)^k 945z^{-1}r + 945$
7	$(-1)^k (784r^3 + r^7 - 56r^5 - 2304r)z^{-7} + (10612r^2 - 1190r^4 - 11025 + 28r^6)z^{-6} + (-1)^k (53487r + 378r^5 - 11340r^3)z^{-5} + (99225 - 59850r^2 + 3150r^4)z^{-4} + (-1)^k (17325r^3 - 173250r)z^{-3} + (-218295 + 62370r^2)z^{-2} + (-1)^k 135135z^{-1}r + 135135$
9	$(-1)^k (-52480r^3 + 147456r + r^9 - 120r^7 + 4368r^5)z^{-9} + (45r^8 + 120330r^4 - 909765r^2 - 4410r^6 + 893025)z^{-8} + (-1)^k (-5742495r - 76230r^5 + 1451835r^3 + 990r^7)z^{-7} + (-13097700 + 9514890r^2 - 796950r^4 + 13860r^6)z^{-6} + (-1)^k (33648615r - 5405400r^3 + 135135r^5)z^{-5} + (-23648625r^2 + 51081030 + 945945r^4)z^{-4} + (-1)^k (-61486425r + 4729725r^3)z^{-3} + (16216200r^2 - 72972900)z^{-2} + (-1)^k 34459425z^{-1}r + 34459425$

approximations fitting the ideal responses in a wide range of frequencies, in both magnitude and phase, can be observed. In Figure 8b it can be observed that the approximations fulfill the two desired properties: (i) all the poles and zeros lie inside the unit circle, and (ii) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$.

8. CONCLUSIONS

We have presented a new IIR (infinite impulse response) type digital fractional order differentiator (DFOD) with a tuning knob to compromise the high frequency approximation accuracy. It is a direct discretization of fractional order differentiators in continuous time domain. The basic idea is the use of a new family of first order digital differentiators expressed in the second order IIR filter form [2] via the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule.

The MATLAB codes developed in this paper are available from the authors upon email request.

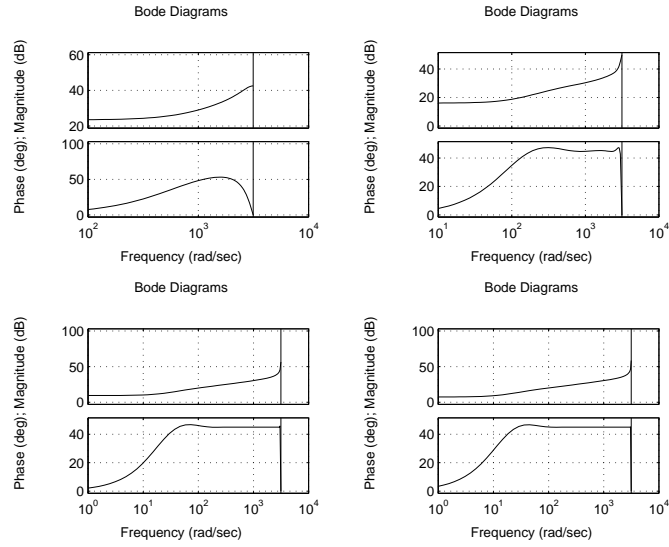


Figure 7. Bode plots (approximation orders 1,3,7,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

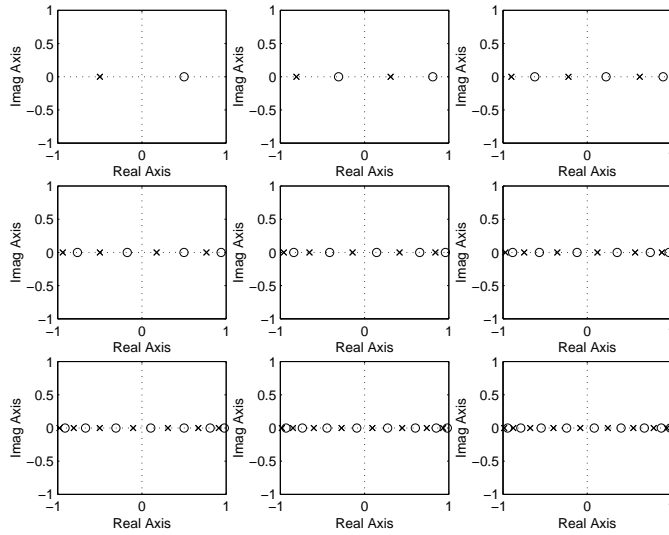


Figure 8. Zero-pole distribution (approximation orders 1,2,...,9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

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