Identifying Diffusion-Wave Constant, Fractional Order And Boundary Profile Of A Time-Fractional Diffusion-Wave Equation From Noisy Boundary Measurements *

Jinsong Liang[1], YangQuan Chen[1], Blas M. Vinagre[2] and Igor Podlubny[3]

[1]Center for Self-Organizing & Intelligent Systems (CSOIS),
Dept. of Electrical & Computer Engineering,
Utah State University, 4160 Old Main Hill, Logan, UT 84322-4160, USA.
{jsliang, yqchen}@ieee.org

[2]Dept. of Electronic and Electromechanical Engineering, Industrial Engineering School,
University of Extremadura, 06071-Badajoz, Spain
bvinagre@unex.es

[3]Dept. of Informatics and Process Control,
Technical University of Kosice, 042 00 Kosice, Slovak Republic
igor.podlubny@tuke.sk

Abstract — *In this paper, a hybrid symbolic and numerical method is used to identify the unknown wave constant, fractional order and initial profile of a fractional order diffusion-wave equation based on boundary measurements. The unknown initial profile is parameterized by a polynomial. The measurement noise is also considered. Two extreme cases are also tested to show the robustness of the proposed identification method. The effectiveness and advantages are demonstrated via simulation examples.*

1 Introduction

In this paper, a hybrid symbolic and numerical method is used to identify the unknown wave constant, fractional order and initial profile of a fractional order diffusion-wave equation based on boundary measurements possibly corrupted with measurement noise. Fractional wave equations are obtained from the classical diffusion (resp. wave) equations by replacing the first (resp. second) order time derivative term by a fractional-order

*Corresponding author's email: yqchen@ece.usu.edu; tel: (435)797-0148; fax: (435)797-0354; URL: http://www.csois.usu.edu/people/yqchen

derivative with order $1 < \alpha < 2$. Since many of the universal phenomenons can be modelled accurately using the fractional diffusion-wave equations (see [1]), there has been a growing interest in investigating the solutions and properties of these equations. Research has been focused on the analytical solution to the fractional diffusion and wave equations (see [2], [3], [4], and [5]). Compared with the publications on the control of the wave equation (see [6], [7], [8], and [9]), research results on the control of fractional wave equations are very few (see [10]). In control engineering, system identification and parameter estimation play an important role, because the more knowledge we have about the plant to be controlled, the better the controller we can design. To the best of the authors' knowledge, this is the first research result on the parameter estimation of systems governed by the fractional diffusion-wave equations. We will demonstrate how we can identify the unknown diffusion-wave constant, unknown fractional order and the unknown initial boundary profile simultaneously from the boundary measurements.

The paper, an extended version of [11], is organized as follows. In Sec. 2, the problem formulation is given. In Sec. 3, the system identification method used in this paper is presented. In Sec. 4, we validate the algorithm via simulation examples. Sec. 5 studies two extreme cases. In Sec. 6, we study the relationship between estimation accuracy and the order of the polynomial used to estimate the initial profile. Finally, Sec. 7 concludes this paper.

2 Problem Formulation

Consider a cable made from special materials, with one end fixed and the other end free, governed by the following fractional diffusion-wave equation

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= b^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \\ & 1 < \alpha < 2,\ x \in [0,1],\ t \geq 0, \end{aligned} \tag{1}$$

where $u(x,t)$ is the displacement of the cable at $x \in [0,1]$ and $t \geq 0$, α is the parameter describing the order of the fractional derivative, b is a constant decided by the tension and the mass per unit length of the cable. Clearly, this is a special type of fractional order PDE system in-between the diffusion equation ($\alpha = 1$) and the wave equation ($\alpha = 2$).

Equation (1) is subject to the following initial and boundary conditions

$$u(0,t) = 0, \tag{2}$$

$$u_x(1,t) = f(t), \tag{3}$$

$$u(x,0) = u_0(x), \tag{4}$$

$$u_t(x,0) = v_0(x), \tag{5}$$

where $f(t)$ is the boundary control force at the free end, $u_0(x)$ and $v_0(x)$ are the initial displacement condition (initial boundary profile) and the initial velocity profile, respectively.

We adopt the following Caputo definition for fractional derivative of order α of any function $f(t)$, because the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations (see [12],[13])

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \tag{6}$$

where n is an integer satisfying $n - 1 < \alpha \leq n$.

In this paper we assume the values of α and b are not exactly known and need to be estimated. Furthermore, initial profiles $u_0(x)$ and $v_0(x)$ may not be exactly known. We also assume that the displacement of the free end $u(1,t)$ can be measured for the identification task.

3 The Proposed Identification Method

For simplicity of our presentation and for practical reasons, we consider a simplified scenario. That is, with the initial shape $u_0(x)$, not exactly known, initial velocity $v_0(x) = 0$ (cable is initially at rest), and no boundary force $f(t) = 0$, we can measure the displacement of the free end $u(1,t)$ as the system output measurement data for identification.

If equations (1)-(5) can be solved, we can estimate α and b through an optimization program to make the solution fit the measurement data as closely as possible.

There are following problems with the above idea.

1. How to solve equations (1)-(5)?
2. Since we are designing the parameter estimation algorithm via simulation, how to generate the measurement data?
3. If we want to do a real hands-on experiment rather than generating the "measured data" via simulation, it is hard to make the initial shape $u(x,0)$ exactly as desired, *i.e.*, the actual initial shape is not exactly known.

The first two problems are very closely related. The first problem can be solved by numerically solving equations (1)-(5), since the analytical solution is still an unsolved problem. We developed a method combining the symbolic computation and numerical computation to solve the fractional diffusion-wave equation, effective even if $f(t)$, a boundary feedback controller, is included. The solution plus Gaussian noise, which is unavoidable in the actual experiments, can be used as the measured data. This is how we solve the second problem. It is illustrated below how to solve (1)-(5), assuming $\alpha = 1.75$, $b = 0.5$, $f(x) = 0$, and the initial conditions

$$u_0(x) = -\frac{1}{2}\sin(\frac{1}{2}\pi x), \tag{7}$$

$$v_0(x) = 0. \tag{8}$$

Based on the definition of (6), the Laplace transform of the fractional derivative is

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+)s^{\alpha-1-k} \tag{9}$$

Taking the Laplace transform of (1)- (3) with respect to t, we obtain the following ODE (Ordinary Differential Equation) and boundary conditions

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} - 4s^{\frac{7}{4}}U(x,s) = 2s^{\frac{3}{4}}\sin(\frac{1}{2}\pi x) \tag{10}$$

$$U(0,s) = 0 \tag{11}$$

$$\frac{\mathrm{d}U(x,s)}{\mathrm{d}x}|_{x=1} = 0 \tag{12}$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$.

Solving (10), we have

$$\begin{aligned} U(x,s) &= e^{-2\,s^{\frac{7}{8}}x}C_1 + e^{2\,s^{\frac{7}{8}}x}C_2 \\ &\quad -8\,\frac{s^{3/4}\sin\left(1/2\,\pi\,x\right)}{16\,s^{7/4}+\pi^2}) \end{aligned} \tag{13}$$

where C_1 and C_2 are arbitrary constants. By taking derivative of (13) with respect to x, we have

$$\begin{aligned} \frac{\mathrm{d}U(x,s)}{\mathrm{d}x} &= -2\,s^{\frac{7}{8}}e^{-2s^{\frac{7}{8}}x}C_1 + 2\,s^{\frac{7}{8}}e^{2s^{\frac{7}{8}}x}C_2 \\ &\quad -\frac{4s^{\frac{3}{4}}\cos\left(\frac{1}{2}\,\pi\,x\right)\pi}{16\,s^{\frac{7}{4}}+\pi^2}. \end{aligned} \tag{14}$$

Substituting (13) and (14) into (11) and (12), respectively, yields

$$C_1 + C_2 = 0, \tag{15}$$

$$-e^{-2s^{\frac{7}{8}}}C_1 + e^{2s^{\frac{7}{8}}}C_2 = 0. \tag{16}$$

Solving (15) and (16) simultaneously, we obtain

$$C_1 = C_2 = 0. \tag{17}$$

So, finally,

$$U(x,s) = -8\,\frac{s^{3/4}\sin\left(1/2\,\pi\,x\right)}{16\,s^{7/4}+\pi^2}. \tag{18}$$

So far, all the calculation steps illustrated above can be automated via computer symbolic algebra, such as Matlab Symbolic Math Toolbox [14]. Now, $u(x,t)$ can be obtained by taking the numerical inverse Laplace transform of (18). Among the existing numeric inverse Laplace transform methods, the FFT (Fast Fourier Transform) method is both accurate and fast (see [15]). We choose the program in (see [16]) to take the inverse Laplace transform of $U(x,s)$.

We can solve the third problem by treating the initial shape $u_0(x)$ as the extra parameters to be estimated such that the parameter estimation algorithm does not depend on the exact knowledge of $u_0(x)$. Specifically, we assume the initial shape to be parameterized by the following polynomial, which equals to zero at $x = 0$,

$$\tilde{u}_0(x) = \sum_{n=1}^{N} a_n x^n \tag{19}$$

where a_i is the parameter to be estimated. By increasing N, we expect that the estimated initial shape $\tilde{u}_o(x)$ will converge to $u_0(x)$, the real initial shape.

Now the parameter estimation problem can be formulate as the following nonlinear programming problem

$$\begin{aligned} \min_{a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}} J(a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}) &= \\ \min_{a_0,\cdots,a_N,\tilde{\alpha},\tilde{b}} \sum_{n=0}^{N_s-1} \left(u(1,n\Delta t) - \tilde{u}(1,n\Delta t)\right)^2, \end{aligned} \tag{20}$$

where $a_0,\ldots,a_N,\tilde{\alpha},\tilde{b}$ are the parameters to be estimated; $u(1,n\Delta t)$ are the measured boundary response data at time $n\Delta t$ with the sampling time Δt; $\tilde{u}(1,n\Delta t)$ are the solution to (1)-(5) based on parameters $a_0,\ldots,a_N,\tilde{\alpha},\tilde{b}$ and N_s is the total number of samples.

At this step, the identification problem has been converted to a numerical optimization problem which can be solved by various existing optimization codes. The optimization program we chose for this study is `SolvOpt` (see [17]), a free program for local nonlinear optimization problems.

Another source of errors in this algorithm is from the mismatched time between the measured data and the numerical solution. In (20), $u(1,n\Delta t)$ is desired to be measured at $t = n\Delta t$. However, due to various reasons in practice, especially the inaccuracy of the starting time, the actual time at which $u(1,t)$ is sampled can be slightly different from the desired time instant $n\Delta t$. We simulated the effect of this time mismatch problem by using the following objective function, a slightly modified one from (20):

$$\begin{aligned} \min J(a_0,\ldots,a_N,\tilde{\alpha},\tilde{b}) = \sum_{n=0}^{N_s-k-1} & \left(u(1,(n+k)\Delta t)\right. \\ & \left. -\tilde{u}(1,n\Delta t)\right)^2, \end{aligned} \tag{21}$$

i.e., the simulated mismatched time is $k\Delta t$.

4 Simulation Results for Algorithm Validation

To generate the simulated measurement data, the following parameters and initial conditions are used

$$\begin{aligned} \alpha = 1.75,\ b = 0.25,\ \Delta t = 0.078s,\ N_s = 512 \\ u_0(x) = -\frac{1}{2}\sin(\frac{1}{2}\pi x) \\ v_0(x) = 0. \end{aligned} \tag{22}$$

The corresponding solution for $u(x,t)$ obtained from (1)-(5) is plotted in Fig. 1. From Fig. 1 we can see that the fractional diffusion-wave equation have mixed properties of

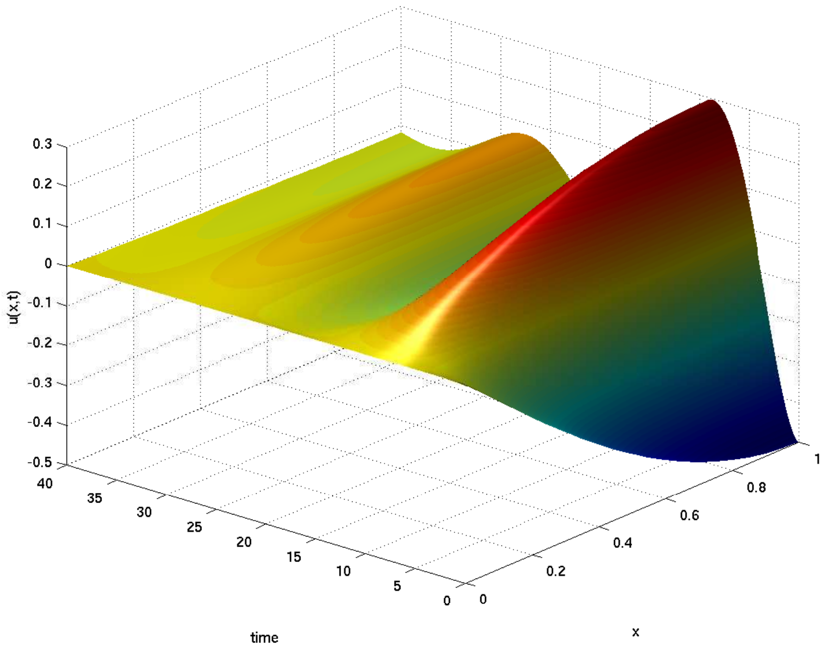

Figure 1: Displacement of the whole cable $u(x,t)$

the diffusion equation, *i.e.*, damped solution, and of the wave equation, *i.e.*, oscillated solution.

To simulate the practical measured signal, Gaussian noise with $SNR = 26dB$ is added to the displacement of the free end and shown in Fig. 2.

We choose $N = 3$ and the parameters to be estimated are initialized in optimization as follows

$$a_0 = -0.79,\ a_1 = 0.07,\ a_2 = 0.21, \tag{23}$$

and

$$\tilde{\alpha} = 1.5,\ \tilde{b} = 1. \tag{24}$$

In (23), a_0, a_1, and a_2 are initialized to make the polynomial (19) close to (22). This is reasonable since the actual initial profile is actually roughly known.

We used different values of k to simulate different amount of mismatched time. After the optimization process, the estimated values of α and b are shown in Table 1. The estimated initial shapes are plotted in Fig. 3.

We can see that the unknown system parameters and the initial profile have been successfully estimated. As expected, smaller mismatched time generates more accurate results. However, for relatively large mismatched time, the estimation accuracy is still satisfactory. In the sequel, we only report the results for noisy measurement cases if not otherwise stated.

5 Simulation Studies on Two Extreme Cases

In this section, we study two extreme cases, i.e., when α is close to 1 and when $\alpha = 2$. In many existing schemes, in extreme cases, the estimation accuracy is usually degraded,

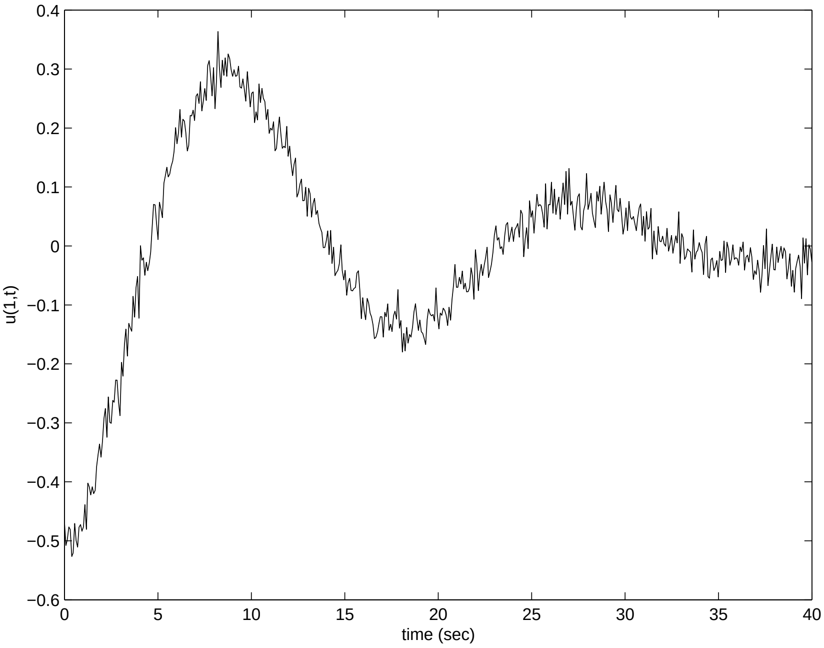

Figure 2: Boundary measurement data with Gaussian noise added, $SNR = 20dB$

Table 1: Estimated parameters, $\alpha = 1.754$, $b = 0.25$, $N = 3$

	time mismatch	$\tilde{\alpha}$ and rel. error	$\tilde{b}$ and rel. error
$k = 0$	$0s$	$1.7520, 0.11\%$	$0.2497, 0.12\%$
$k = 1$	$0.078s$	$1.7557, 0.32\%$	$0.2504, 0.16\%$
$k = 2$	$0.156s$	$1.7582, 0.46\%$	$0.2513, 0.52\%$
$k = 5$	$0.390s$	$1.7655, 0.88\%$	$0.2539, 1.56\%$

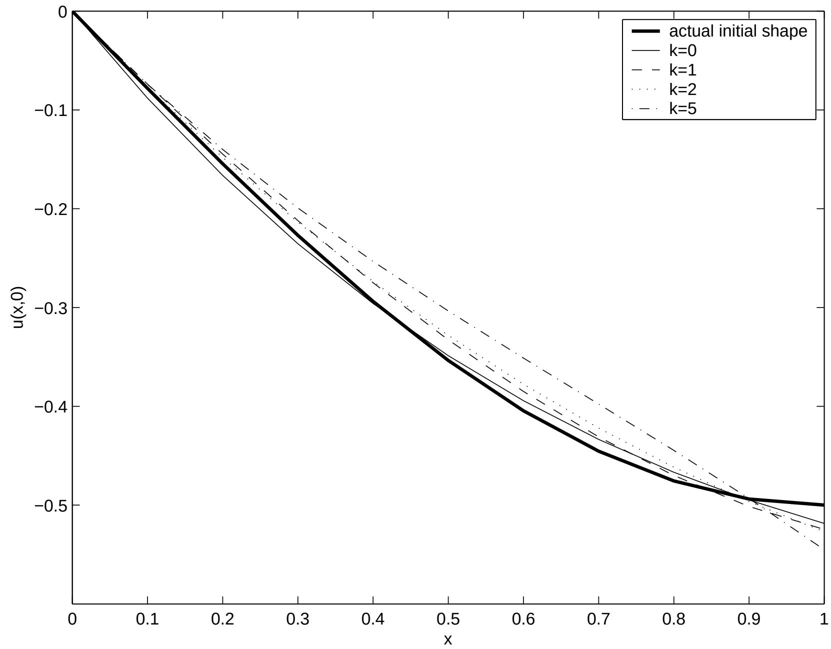

Figure 3: Actual initial shape and estimated initial shape, $\alpha = 1.75$

Table 2: Estimated parameters when $\alpha = 1.1$, $b = 0.25$, $N = 3$

	time mismatch	$\tilde{\alpha}$ and rel. error	$\tilde{b}$ and rel. error
$k = 0$	$0s$	$1.1011, 0.10\%$	$0.2483, 0.68\%$
$k = 1$	$0.078s$	$1.1060, 0.54\%$	$0.2461, 1.56\%$
$k = 2$	$0.156s$	$1.1064, 0.58\%$	$0.2463, 1.48\%$
$k = 5$	$0.390s$	$1.1185, 1.68\%$	$0.2398, 4.08\%$

or even worse, the algorithm may fail. It is meaningful to check the robustness of our proposed algorithms in these extreme cases.

First let us study the parameter estimation when $\alpha = 1.1$. In this case, the fractional wave equation is closer to a diffusion equation than to a wave equation. All parameters and initial conditions are the same as in Sec. 4 except α, the fractional order.

The solution to (1)-(5), to be taken as measurement data for system identification, is plotted in Fig. 4. We can see that the solution is over-damped, close to the solution of the diffusion equation.

The boundary measurement data with Gaussian noise added is plotted in Fig. 5.

The estimated parameters are shown in Table 2. The estimated initial shapes are plotted in Fig. 6.

Simulation results show that the algorithm works well even if α is close to 1. However, comparing Tbl. 2 and Tbl. 1, we can see that the estimation accuracy degraded when α is close to 1. We can see the reason if we study Fig. 5, where after $t = 20$sec, $u(1, t)$ is

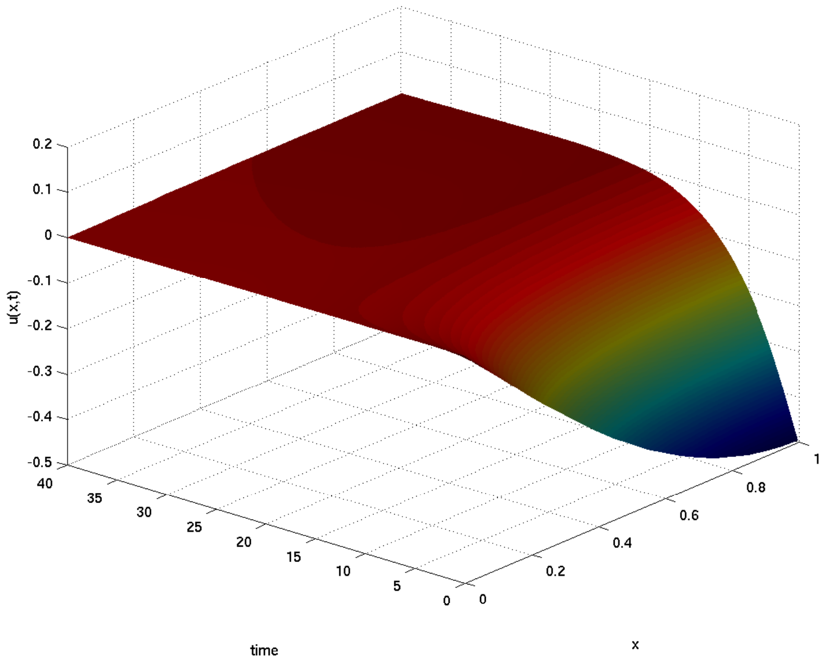

Figure 4: Displacement of the whole cable $u(x,t)$ when $\alpha = 1.01$

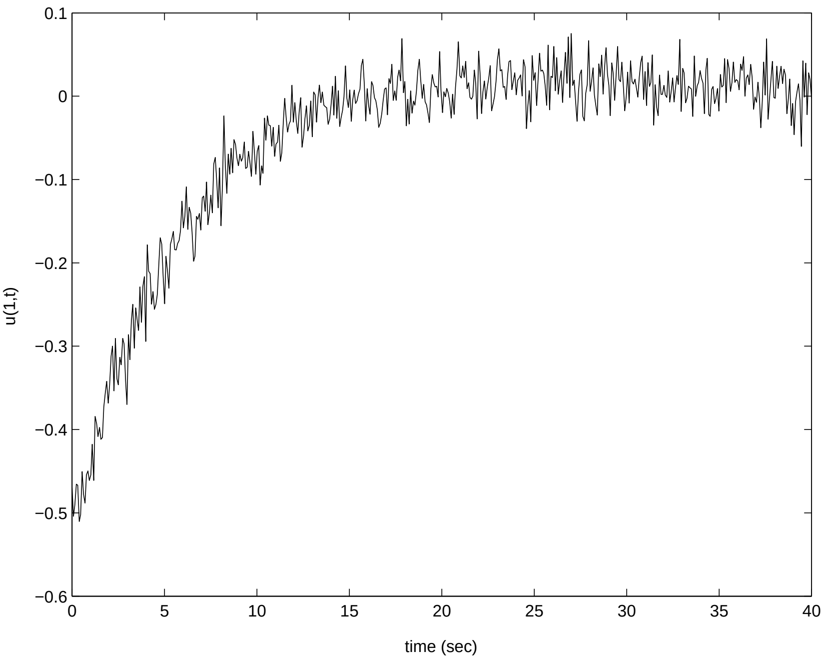

Figure 5: Boundary measurement data for identification with Gaussian noise added when $\alpha = 1.1$, $SNR = 26$dB

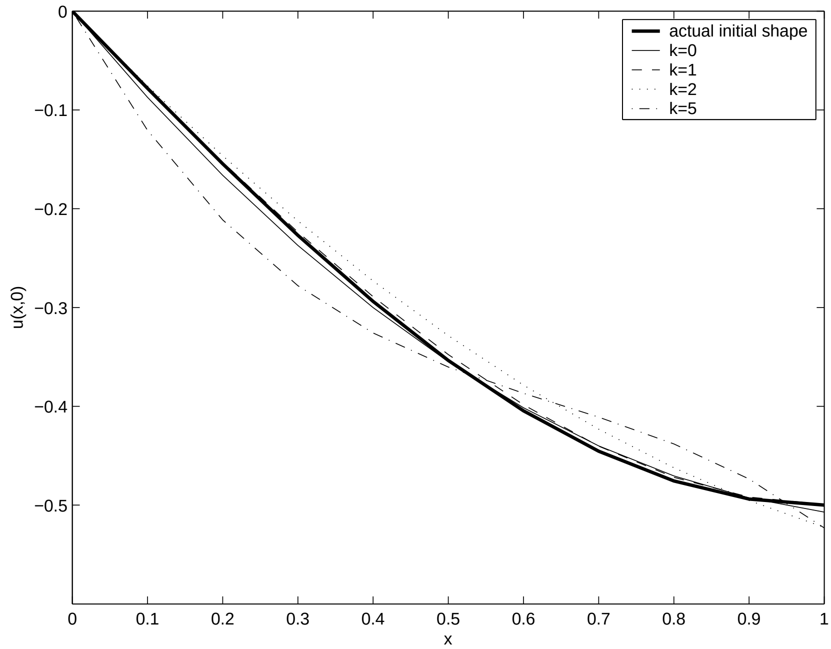

Figure 6: Actual initial shape and estimated initial shape, $\alpha = 1.1$

Table 3: Estimated parameters when $\alpha = 2$, $b = 0.25$, $N = 3$

	time mismatch	$\tilde{\alpha}$ and rel. error	$\tilde{b}$ and rel. error
$k = 0$	$0s$	$2.0001, 0.00\%$	$0.2500, 0.00\%$
$k = 1$	$0.078s$	$2.0002, 0.01\%$	$0.2507, 0.28\%$
$k = 2$	$0.156s$	$2.0004, 0.02\%$	$0.2514, 0.56\%$
$k = 5$	$0.390s$	$2.0006, 1.63\%$	$0.2537, 1.48\%$

almost zero and there is only noise left. So Fig. 5 actually contains less useful information than in Fig. 2, making the estimation accuracy lower.

Next we study the parameter estimation when $\alpha = 2$, *i.e.*, the fractional wave equation becomes the wave equation.

The solution to (1)-(5) is plotted in Fig. 7. This is the solution to the standard wave equation

The boundary measurement data with Gaussian noise added is plotted in Fig. 8. The estimated parameters are shown in Tbl. 3 and the estimated initial shapes are plotted in Fig. 9.

In the case of $\alpha = 2$, the algorithm works even better than in the case of $\alpha = 1.75$. In Fig. 9, the estimated initial shapes are almost identical to the actual initial shape, even if the mismatched time is large. This confirms our reasoning why the estimation results for $\alpha = 1.75$ are better than the results for $\alpha = 1.1$. Since the solution for the standard wave equation oscillates forever without being damped, Fig. 8 contains more useful information

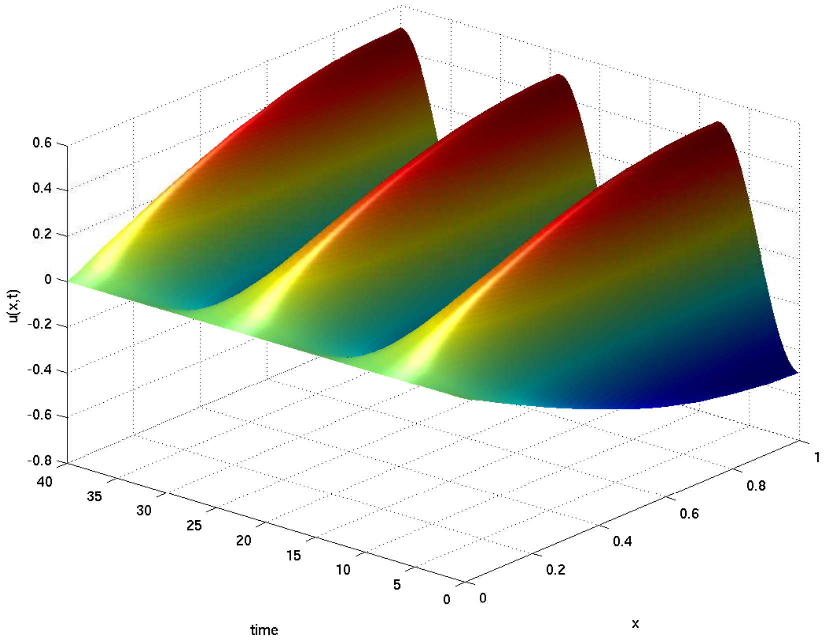

Figure 7: Displacement of whole cable when $\alpha = 2$

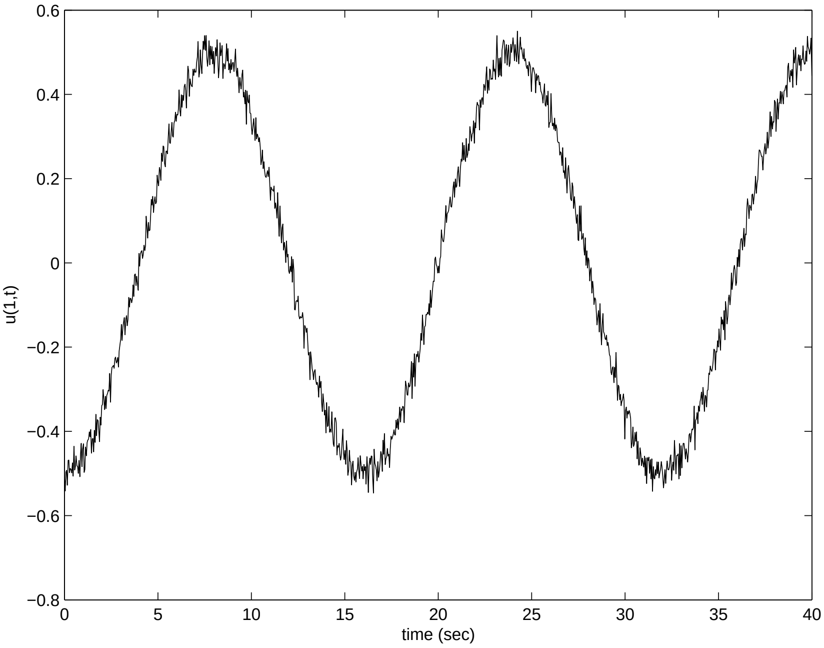

Figure 8: Boundary measurement data with Gaussian noise added when $\alpha = 2$, $SNR = 26dB$

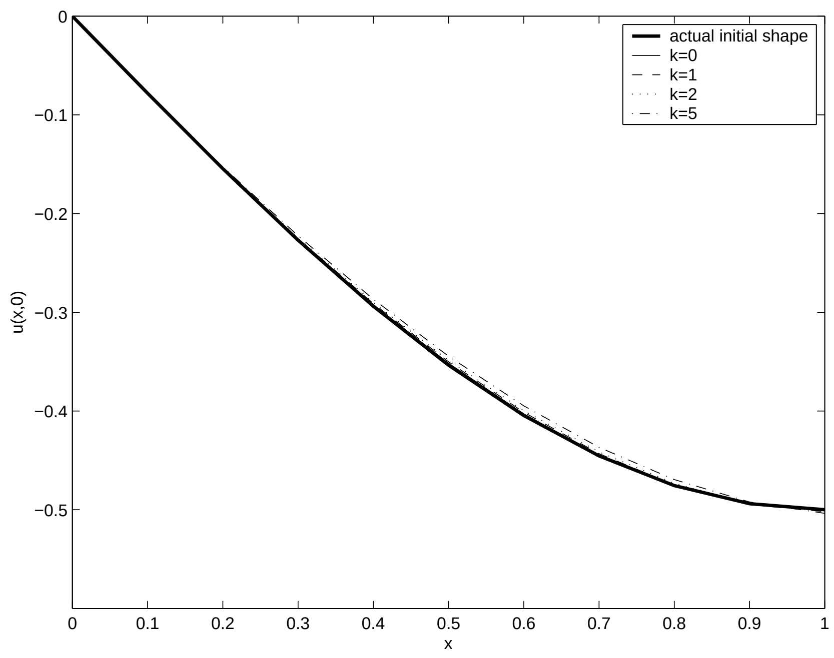

Figure 9: Actual initial shape and estimated initial shape, $\alpha = 2$

than in Fig. 2, which leads to better estimation results.

It can be concluded that even in the extreme cases, the performance are still satisfactory by using our proposed identification algorithm.

6 Relationship between polynomial order and estimation accuracy

In the previous simulation examples, we use the third order polynomial to estimate the initial profile, which leads to satisfactory estimation results. In this section, we will investigate the relation between the polynomial order and the estimation accuracy. Lower order polynomial results in less parameters to identify and faster computation, which is important for tasks requiring on-line parameter estimation . So lower order polynomial is highly preferred if it generates equally high, or lower yet acceptable, estimation accuracy.

To study only the relationship between the polynomial order and the estimation accuracy, in the following simulations, all parameters and initial conditions are the same as in Sec. 4, except that there is no measurement noise and no mismatched time ($k = 0$).

We tested three different cases: $N = 3$, $N = 2$, and $N = 1$. The estimated parameters for each case are listed in Tbl. 4. The estimated initial profiles for each case are plotted in Fig. 10. Since we assume there is no noise and mismatched time in the simulations, the estimation accuracy in the $N = 3$ case is mush higher than in the simulations in previous sections. We can see that the estimation accuracy for $N = 2$ is almost as high as for $N = 3$. We conclude that unless extremely high estimation accuracy is required, the second order polynomial can replace the third order polynomial to estimate the initial profile. The first order polynomial is also a candidate when the measurement noise is small, the mismatched time is small, and the estimation accuracy requirement is low. We

Table 4: Estimated parameters when $\alpha = 1.75$, $b = 0.25$, different Ns

	$\tilde{\alpha}$ and rel. error	$\tilde{b}$ and rel. error
$N = 3$	$1.7500, 0.00\%$	$0.2500, 0.00\%$
$N = 2$	$1.7501, 0.006\%$	$0.2500, 0.00\%$
$N = 1$	$1.7543, 0.24\%$	$0.2491, 0.36\%$

can also conclude that this estimation algorithm is not sensitive to the difference between the actual initial profile and the desired initial profile.

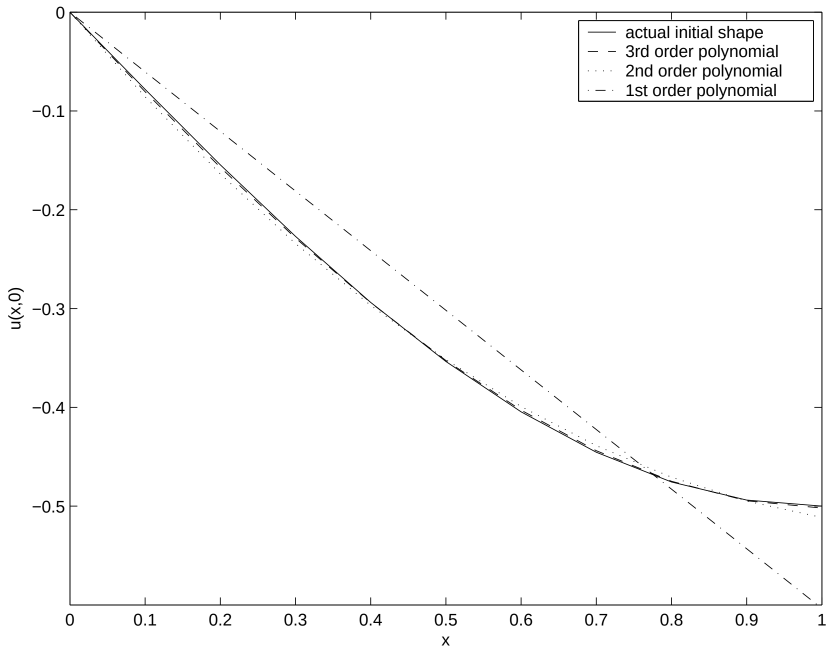

Figure 10: Actual initial shape and estimated initial shape, different Ns, $\alpha = 1.75$

7 Concluding Remarks

Simulation results show the effectiveness of the parameter estimation algorithms proposed in this paper. The algorithm does not rely on the exact knowledge of the actual initial condition and is insensitive to the difference between the actual initial shape in the experiment and the desired initial shape. The accuracy is satisfactory even when facing the relative large time mismatch and extreme cases for fractional order. The presented algorithm is expected to work for the parameter estimation of the temporal-spatial fractional diffusion equations given by $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b^2 \frac{\partial^\beta u(x,t)}{\partial x^\beta}$ with positive real numbers α and β, which will be our future research efforts.

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A Source code symbolically solving the time-fractional diffusion-wave equation

Following is a piece of example code for symbolically solving the fractional diffusion-wave equation with $\alpha = 1.75$ and $b = 0.25$.

```
clear all
syms x;
PI = sym(pi);
torder = 1.75;
b_sqr = 0.25^2;
u_0 = -0.5*sin(PI/2*x);
v_0=0;
ode = strcat(char(sym(b_sqr)), '*D2U', ...
      '-', char(s^torder), '*U', '+(', ...
      char(s^(torder-1)*sym(u_0)),')+(', ...
      char(s^(torder-2)*sym(v_0)), ')',' = 0');
U_ud = simple(dsolve(ode, 'x'));
dU_ud = simple(diff(U_ud, 'x', 1));
bd1 = strcat(char(subs(U_ud, x, 0)), '=0');
bd2 = subs(dU_ud, x, 1);
bd2 = strcat(char(bd2), '=0');
[C1, C2] = solve(bd1, bd2, 'C1', 'C2');
U_xs = subs(U_ud);
```

About the Authors

Jinsong Liang received his Ph.D. from the Mechanical and Aerospace Engineering Dept. of Utah State University in 2005 and he is presently a M.Sc. candidate at the Electrical and Computer Engineering Department at Utah State University.

YangQuan Chen is presently an assistant professor of Electrical and Computer Engineering Dept. and the Acting Director for CSOIS (Center for Self-Organizing & Intelligent Systems, `www.csois.usu.edu`) at Utah State University. He obtained his Ph.D. from Nanyang Tech. Univ. (NTU), Singapore in 1998. His current research interests include fractional order dynamic systems and control, distributed parameter systems and control, mobile actuator and sensor networks, iterative learning control and interval computation.

Blas M. Vinagre received his M.Sc. degree in Telecommunications Engineering and the Ph.D. degree in Industrial Engineering in 1985 and 2001, respectively. His research focuses on control theory and applications of fractional calculus in control, robotics and signal processing.

Igor Podlubny is now the Head of the Dept. of Informatization and Control of Processes, Faculty BERG, Technical University of Kosice, Slovak Republic. He obtained his University Professor title in 2001 awarded by the President of the Slovak Republic and his University Docent habilitation in 1995 in process control, Technical University of Kosice. He received his PhD in 1989 in differential equations and mathematical physics from Odessa State University, Odessa, Ukraine (former USSR).