

# Iterative Learning Control of Uncertain Discrete-time Nonlinear Feedback Systems With A Control Saturater \*

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**Abstract** — *A simple iterative learning controller (ILC) together with a feedback controller is proposed for the tracking control of uncertain discrete-time nonlinear systems. A uniform bound of the tracking error is obtained in the presence of bounded uncertainty, disturbance and the re-initialization error. Control input saturation is also considered. Simulation illustrations are given to show the effectiveness of the proposed scheme.*

**Keywords:** Iterative learning control; Repetitive systems; Nonlinear uncertain systems; Discrete-time nonlinear system; Re-initialization error; Control saturation.

## 1 Introduction

Iterative Learning Control (ILC)<sup>1</sup> has drawn increasing attentions for its simple rationale that the control performance improvement can be obtained from system's repetitive operations [1, 2, 3, 4]. Usually, ILC algorithms are implemented in digital form. So, the analysis of discrete-time ILC is important. Some results have been obtained for the linear discrete-time systems [5, 6, 7]. Nonlinear discrete-time systems were considered in [8, 9, 10]. The robustness analysis for time-varying uncertain discrete-time systems were considered by Saab [7, 11].

A feedback controller can either improve the transient performance of open-loop systems or can stabilize unstable systems. As a result, better ILC performance can be

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achieved by introducing a feedback loop. Then the system considered is actually controlled by an ILC controller in the iteration number direction and a feedback controller in the time direction simultaneously. Under this framework, the ILC convergence results for continuous-time nonlinear systems were established in [13, 14, 15]. In the discrete-time case, however, no convergence and robustness result is available for ILC schemes when a feedback controller is included. This paper is to solve the problem by studying a simple ILC updating law together with a feedback controller. A uniform bound of the tracking error is obtained in the presence of bounded uncertainty, disturbance and the re-initialization error. It is shown in this paper that the final tracking error bound is a class- $K$  function [16] of the bounds of uncertainty, disturbance and the re-initialization error. In practice, control input may be saturated. Therefore, it is also important to consider this problem in theoretical analysis. In [17], the saturation is considered as a box operator to address the bounded input in iterative learning control. In a different way, the analysis of this paper verifies the observation in [17] that the process of moving an input back to a bound if it exceeds it does not affect the contraction mapping property.

The organization of this paper is as follows. In Sec. 2, the control problem is formulated with the proposed ILC scheme together with a feedback controller. Some preliminaries for the robust ILC convergence analysis are given in Sec. 3. The main result of the ILC convergence and robustness is given in Sec. 4. The detailed proof is also given. Some simulation illustrations are presented in Sec. 5 to verify the effectiveness of the proposed schemes. Finally, conclusions are drawn in Sec. 6.

The norms used in this paper are defined as follows:

$$\|\bar{v}\| = \max_{1 \leq i \leq n} |\bar{v}_i|, \quad \|G\| = \max_{1 \leq i \leq m} \left( \sum_{j=1}^n |g_{i,j}| \right)$$

where  $\bar{v} = [\bar{v}_1, \dots, \bar{v}_n]^T$  is a vector,  $G = [g_{i,j}]_{m \times n}$  is a matrix.

## 2 Problem Statement

Consider the following discrete-time uncertain nonlinear time-varying system:

$$\begin{cases} x_i(t+1) = f(x_i(t), t) + B(x_i(t), t)u_i(t) + w_i(t) \\ y_i(t) = C(t)x_i(t) + v_i(t) \end{cases} \quad (1)$$

where  $i$  denotes the  $i$ -th repetitive operation of the system;  $t \in \{0, 1, \dots, N\}$ ;  $x_i(t) \in R^n$ ,  $u_i(t) \in R^m$ , and  $y_i(t) \in R^r$  are the state, control input, and output of the system, respectively;  $C(t) \in R^{r \times n}$  is a time-varying matrix; the functions  $f(\cdot, \cdot) : R^n \times [0, N] \mapsto R^n$  and  $B(\cdot, \cdot) : R^n \times [0, N] \mapsto R^m$  are uniformly globally Lipschitz in  $x$ , i.e.,  $\forall t \in [0, N], \forall i, \exists$  constants  $k_f, k_B$ , such that

$$\|\Delta f_i(t)\| \leq k_f \|\Delta x_i(t)\|, \quad \|\Delta B_i(t)\| \leq k_B \|\Delta x_i(t)\|$$

where  $\Delta f_i(t) \triangleq f(x_i(t), t) - f(x_{i-1}(t), t)$ ,  $\Delta B_i(t) \triangleq B(x_i(t), t) - B(x_{i-1}(t), t)$ ,  $\Delta x_i(t) \triangleq x_i(t) - x_{i-1}(t)$ ;  $w_i(t), v_i(t)$  are uncertainty or disturbance to the system bounded with unknown bounds  $b_w, b_v$  defined as

$$b_w \triangleq \sup_{t \in [0, N]} \|w_i(t)\|, \quad b_v \triangleq \sup_{t \in [0, N]} \|v_i(t)\|, \quad \forall i. \quad (2)$$

The problem is formulated as follows. Given a desired trajectory  $y_d(t)$ , starting from an arbitrary initial control input  $u_0(t)$ , obtain the next control input  $u_1(t)$  and the subsequent series  $\{u_i(t) \mid i = 2, 3, \dots\}$  for system (1) by using a proper learning control updating law in such a way that when  $i \rightarrow \infty, y_i(t) \rightarrow y_d(t) \pm \varepsilon^*$  in the presence of bounded uncertainty, disturbance and re-initialization error.

We propose an iterative learning controller together with a feedback controller to solve the above problem as follows. At the  $i$ -th ILC iteration, the control input  $u_i(t)$  to the system (1) is the output of a saturator, i.e.,

$$u_i(t) = \text{sat}(\bar{u}_i(t)) \quad (3)$$

where  $\text{sat}(\bar{u}_i(t)) \triangleq [\text{sat}(\bar{u}_{1_i}(t)), \dots, \text{sat}(\bar{u}_{m_i}(t))]^T$  and

$$\text{sat}(\bar{u}_{j_i}(t)) \triangleq \begin{cases} \bar{u}_{j_i}(t), & \text{if } |\bar{u}_{j_i}(t)| \leq \bar{u}_j^* \\ \frac{\bar{u}_{j_i}(t)}{|\bar{u}_{j_i}(t)|} \bar{u}_j^*, & \text{if } |\bar{u}_{j_i}(t)| > \bar{u}_j^* \end{cases} \quad (4)$$

where  $j = 1, 2, \dots, m$  and the saturation bounds  $\bar{u}_j^* > 0$ . The saturator input is that

$$\bar{u}_i(t) = u_i^f(t) + u_i^b(t) \quad (5)$$

where  $u_i^f(t)$  and  $u_i^b(t)$  are the outputs of the feedforward iterative learning controller and the feedback stabilizing controller, respectively. The overall configuration is shown in Fig. 1.

The feedback stabilizing controller is assumed to be in the following general form.

$$z_i(t+1) = h_a(z_i(t)) + H_b(z_i(t))e_i(t), \quad (6)$$

$$u_i^b(t) = h_c(z_i(t)) + H_d(z_i(t))e_i(t) \quad (7)$$

where  $e_i(t) \triangleq y_d(t) - y_i(t)$  is the output tracking error,  $z_i(t) \in R^{n_c}$  is the state of the feedback stabilizing controller. Without loss of generality, assume  $z_i(0) = 0, \forall i$ . The vector-valued functions  $h_a(\cdot) : R^{n_c} \mapsto R^{n_c}$  and  $h_c(\cdot) : R^{n_c} \mapsto R^m$  are designed to be sector-bounded as

$$\|h_a(z_i(t))\| \leq b_{h_a} \|z_i(t)\|, \quad \|h_c(z_i(t))\| \leq b_{h_c} \|z_i(t)\|.$$

The function matrices  $H_b(\cdot) : R^{n_c} \mapsto R^{n_c \times r}$  and  $H_d(\cdot) : R^{n_c} \mapsto R^{m \times r}$  are designed to be uniformly bounded, i.e.,  $\forall t \in [0, N], \forall z_i(t) \in R^{n_c}$ ,

$$\|H_b(z_i(t))\| \leq b_{H_b}, \quad \|H_d(z_i(t))\| \leq b_{H_d},$$

where  $b_{h_a}, b_{h_c}, b_{H_b}, b_{H_d}$  are positive constants which may be unknown. An ILC updating law is used, i.e.,

$$u_{i+1}^f(t) = u_i(t) + Q(t)e_i(t+1) + \bar{Q}(t)e_i(t) \quad (8)$$

where  $\bar{Q}(t)$  and  $Q(t)$  are  $m \times r$  learning matrices which are to be determined to ensure the ILC convergence.

The following assumptions are imposed:

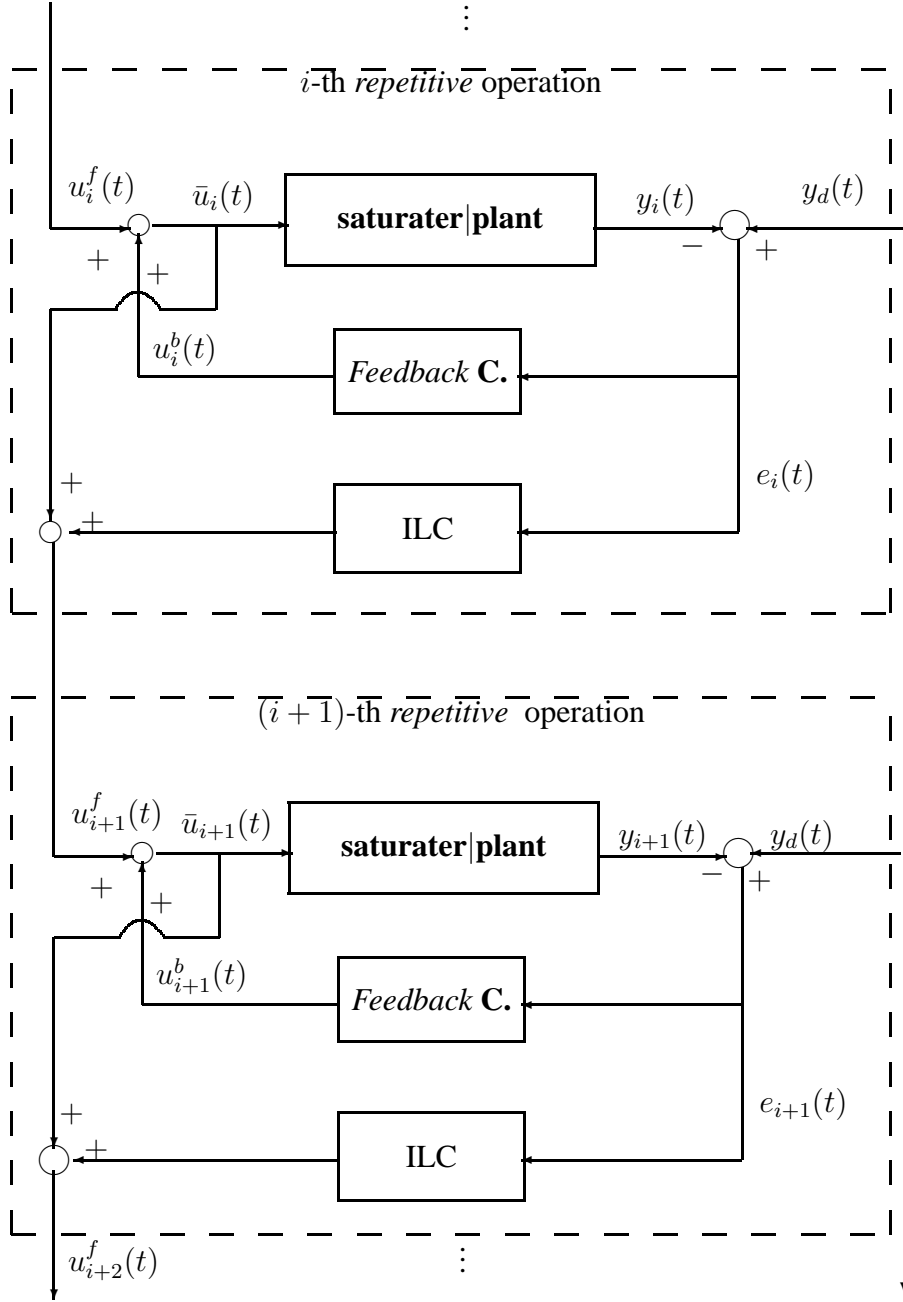


Figure 1: Block-diagram of *iterative learning control* for system with a feedback controller

- A1). The re-initialization error is bounded as follows,  $\forall t \in [0, N], \forall i, \|x_d(0) - x_i(0)\| \leq b_{x_0}, \|y_d(0) - y_i(0)\| \leq b_C b_{x_0} + b_v$ , where  $b_C \triangleq \sup_{t \in [0, N]} \|C(t)\|$ .
- A2). Matrix  $C(\cdot)B(\cdot, \cdot)$  has a full column rank  $\forall t \in [0, N], x(t) \in R^n$ .
- A3). Operator  $B(\cdot, \cdot)$  is bounded, i.e., there exists a constant  $b_B$  such that for all  $i$ ,

$$\sup_{t \in [0, N]} \|B_i(t)\| \leq b_B.$$

where  $B_i(t) \triangleq B(x_i(t), t)$  for brevity.

A4). There exists a unique bounded desired input  $u_d(t)$ ,  $t \in [0, N]$  such that when  $u(t) = u_d(t)$ , the system has a unique bounded desired state  $x_d(t)$  satisfying

$$\begin{cases} x_d(t+1) = f(x_d(t), t) + B(x_d(t), t)u_d(t) \triangleq f_d + B_d u_d \\ y_d(t) = C(t)x_d(t) \triangleq C(t)x_d. \end{cases} \quad (9)$$

For a given desired output  $y_d(t)$ , the desired input  $u_d(t)$  to achieve  $y_d(t)$  is inside the saturation bounds, i.e.,  $u_d(t) \equiv \text{sat}(u_d(t)), \forall t \in [0, N]$ .

### 3 Preliminaries

To analyze the robust convergence property of the proposed ILC algorithm together with a feedback controller, the following  $\lambda$ -norm is introduced for a discrete-time vector  $h(t), t = 0, 1, \dots, N$ .

$$\|h(t)\|_\lambda \triangleq \sup_{t \in [0, N]} \hat{e}^{-\lambda t} \|h(t)\| \quad (10)$$

where  $\lambda > 0$  when  $\hat{e} > 1$  or  $\lambda < 0$  when  $\hat{e} \in (0, 1)$ . The positive constant  $\hat{e}$  will be specified later. It should be pointed out that the  $\lambda$ -norm used in this paper is equivalent to the infinity-norm [11] because  $\|h(\cdot)\|_\lambda \leq \|h(\cdot)\|_\infty \leq \|h(\cdot)\|_\lambda \hat{e}^{\lambda N}$ , where the infinity-norm  $\|h(t)\|_\infty \triangleq \sup_{t \in [0, N]} \|h(t)\|$ .

To facilitate the later derivations, some basic relations are presented in the following. The main purpose is to explore the relationship between  $(\|\delta x_i(t)\|_\lambda + \|z_i(t)\|_\lambda)$  and  $\|\delta u_i^f(t)\|_\lambda$  where  $\delta u_i^f(t) \triangleq u_d(t) - u_i^f(t)$ ,  $\delta x_i(t) \triangleq x_d(t) - x_i(t)$ . Also, we use the following notations:  $b_{u_d} \triangleq \sup_{t \in [0, N]} \|u_d(t)\|$ ;  $\delta u_i(t) \triangleq u_d(t) - u_i(t)$ ;  $\delta f_i(t) \triangleq f_d - f(x_i(t), t)$ ;  $\delta B_i(t) \triangleq B_d - B_i(t)$ . Then, from (1) and (9), it can be obtained that

$$\delta x_i(t+1) = \delta f_i(t) + \delta B_i(t)u_d + B_i(t)\delta u_i(t) - w_i(t). \quad (11)$$

Taking the norm for (11) yields

$$\|\delta x_i(t+1)\| \leq (k_f + b_{u_d}k_B)\|\delta x_i(t)\| + b_B\|\delta u_i(t)\| + b_w. \quad (12)$$

From (7), it can be seen that

$$\|u_i^b(t)\| \leq b_{h_c}\|z_i(t)\| + b_{H_d}b_C\|\delta x_i(t)\| + b_{H_d}b_v. \quad (13)$$

As  $u_d(t) \equiv \text{sat}(u_d(t))$ , we have

$$\begin{aligned} \|\delta u_i(t)\| &= \|u_d(t) - \text{sat}(u_i^f(t) + u_i^b(t))\| \\ &\leq \|\delta u_i^f(t) - u_i^b(t)\| \leq \|\delta u_i^f(t)\| + \|u_i^b(t)\|. \end{aligned} \quad (14)$$

Then, (12) becomes

$$\begin{aligned} \|\delta x_i(t+1)\| &\leq (k_f + b_{u_d}k_B + b_Bb_{H_d}b_C)\|\delta x_i(t)\| \\ &\quad + b_Bb_{h_c}\|z_i(t)\| + b_B\|\delta u_i^f(t)\| + b_Bb_{H_d}b_v + b_w. \end{aligned} \quad (15)$$

On the other hand, it can be observed from (6) that

$$\|z_i(t+1)\| \leq b_{h_a}\|z_i(t)\| + b_{H_b}b_C\|\delta x_i(t)\| + b_{H_b}b_v. \quad (16)$$

Thus, adding (16) into (15) yields

$$(\|\delta x_i(t+1)\| + \|z_i(t+1)\|) \leq \hat{e}(\|\delta x_i(t)\| + \|z_i(t)\|) + b_B\|\delta u_i^f(t)\| + \hat{e} \quad (17)$$

where

$$\begin{aligned} \hat{e} &\triangleq \max\{k_f + b_{u_d}k_B + b_Bb_{H_d}b_C + b_{H_b}b_C, b_{h_a} + b_Bb_{h_c}\} \neq 1; \\ \hat{e} &\triangleq (b_{H_b} + b_Bb_{H_d})b_v + b_w. \end{aligned}$$

It should be noted that as  $\hat{e}$  is a function of upper bounds shown in the above,  $\hat{e}$  can be chosen not equal to 1. In this paper, without loss of generality, we can suppose that  $\hat{e} > 0$ . Iterating (17), we can get

$$\|\delta x_i(t+1)\| + \|z_i(t+1)\| \leq \hat{e}^{t+1}b_{x_0} + \sum_{j=0}^t \hat{e}^{t-j}(b_B\|\delta u_i^f(j)\| + \hat{e}). \quad (18)$$

The following facts are to be used when taking the  $\lambda$ -norm ( $|\lambda| > 1$ ) operation of (18):

- $\|c\|_\lambda \equiv |c|, \forall c \in R$ ;
- $\forall |\lambda| > 1, \sup_{t \in [0, N]} \hat{e}^{-(\lambda-1)t} = 1$ ;
- $\forall t_1 \in [0, N_1], t_2 \in [0, N_2], \text{ if } 0 \leq N_1 \leq N_2 \leq N, \text{ then } \|\delta h(t_1)\|_\lambda \leq \|\delta h(t_2)\|_\lambda$ .

Taking the  $\lambda$ -norm ( $|\lambda| > 1$ ) operation of (18) gives

$$\begin{aligned} \|\delta x_i(t)\|_\lambda + \|z_i(t)\|_\lambda &= \sup_{t \in [0, N]} \hat{e}^{-\lambda t} \left\{ \hat{e}^t b_{x_0} + \sum_{j=0}^{t-1} \hat{e}^{t-j-1} (b_B \|\delta u_i^f(j)\| + \hat{e}) \right\} \\ &= b_{x_0} + \sup_{t \in [0, N]} \sum_{j=0}^{t-1} \hat{e}^{-\lambda(t-j)} \hat{e}^{t-j-1} \{ \hat{e}^{-\lambda j} b_B \|\delta u_i^f(j)\| \} + c_0 \hat{e} \\ &\leq b_{x_0} + b_B \|\delta u_i^f(t)\|_\lambda \sup_{t \in [0, N]} \sum_{j=0}^{t-1} \hat{e}^{-\lambda(t-j)} \hat{e}^{t-j-1} + c_0 \hat{e} \\ &\leq b_{x_0} + b_B O(|\lambda|^{-1}) \|\delta u_i^f(t)\|_\lambda + c_0 \hat{e} \end{aligned} \quad (19)$$

where

$$O(|\lambda|^{-1}) \triangleq \frac{1 - \hat{e}^{-(\lambda-1)N}}{\hat{e}^\lambda - \hat{e}}, \quad c_0 \triangleq \sup_{t \in [0, N]} \frac{\hat{e}^{-(\lambda-1)t} (1 - \hat{e}^t)}{\hat{e} - 1} \leq 1/(\hat{e} - 1).$$

Furthermore, we denote

$$b_Q \triangleq \sup_{t \in [0, N]} \|Q(t)\|, \quad b_{\bar{Q}} \triangleq \sup_{t \in [0, N]} \|\bar{Q}(t)\|, \quad \rho \triangleq \sup_{t \in [0, N]} \|I_m - Q(t)C(t+1)B_i(t)\|, \quad \forall i.$$

## 4 Main Result

A main result for the learning convergence is presented in the following theorem.

**Theorem 4.1** *Consider the repetitive discrete-time uncertain time-varying nonlinear system (1) satisfying assumptions A1)-A4) which is under the control of the ILC updating law (8) and the feedback controller (6)-(7). Given a desired trajectory  $y_d(t)$  over the fixed time interval  $[0, NT_s]$ , if the condition*

$$\rho < 1, \quad (20)$$

is satisfied, then the  $\lambda$ -norm of the tracking errors  $e_i(t)$ ,  $\delta u_i(t)$ ,  $\delta x_i(t)$  are bounded for all  $i$ . For a sufficiently large  $|\lambda|$ ,  $\forall t \in [0, N]$ ,

$$b_{uf} \triangleq \lim_{i \rightarrow \infty} \|\delta u_i^f(t)\|_\lambda \leq b_{uf}(b_{x_0}, b_w, b_v), \quad (21)$$

$$b_u \triangleq \lim_{i \rightarrow \infty} \|\delta u_i(t)\|_\lambda \leq b_u(b_{x_0}, b_w, b_v), \quad (22)$$

$$b_x \triangleq \lim_{i \rightarrow \infty} \|\delta x_i(t)\|_\lambda \leq b_{x_0} + b_B O(|\lambda|^{-1}) b_{uf} + c_0 \hat{\varepsilon}, \quad (23)$$

$$b_e \triangleq \lim_{i \rightarrow \infty} \|e_i(t)\|_\lambda \leq b_C b_x + b_v. \quad (24)$$

Moreover,  $b_u, b_x, b_e$  are **class-K** functions of  $b_w, b_v, b_{x_0}$ , i.e.,  $b_u, b_x, b_e$  converge uniformly to zero as  $i \rightarrow \infty$  in the absence of uncertainty, disturbance and re-initialization error, i.e., as  $b_w, b_v, b_{x_0} \rightarrow 0$ .

**Proof:** The tracking error at the  $i$ -th repetition is

$$e_i(t) = y_d(t) - y_i(t) = C(t)\delta x_i(t) - v_i(t). \quad (25)$$

Investigating the learning control deviation at the  $(i+1)$ -th repetition  $\delta u_{i+1}^f(t)$  gives

$$\begin{aligned} \delta u_{i+1}^f(t) &= \delta u_i(t) - Q(t)e_i(t+1) \\ &= \delta u_i(t) - Q(t)C(t+1)\delta x_i(t+1) + Q(t)v_i(t+1) \\ &= -\bar{Q}(t)C(t)\delta x_i(t) + \bar{Q}(t)v_i(t). \end{aligned} \quad (26)$$

From (11), (26) can be written as

$$\begin{aligned} \delta u_{i+1}^f(t) &= \delta u_i(t) - Q(t)C(t+1)[\delta f_i(t) + \delta B_i(t)u_d + B_i(t)\delta u_i(t) - w_i(t)] \\ &\quad - \bar{Q}(t)C(t)\delta x_i(t) + \bar{Q}(t)v_i(t) + Q(t)v_i(t+1). \end{aligned} \quad (27)$$

Collecting terms and then performing the norm operation for (27) yield

$$\|\delta u_{i+1}^f(t)\| \leq \rho \|\delta u_i(t)\| + [b_Q b_C (k_f + b_{u_d} k_B) + b_{\bar{Q}} b_C] \|\delta x_i(t)\| + b_Q (b_C b_w + b_v) + b_{\bar{Q}} b_v. \quad (28)$$

Based on (14) and (13), (28) becomes

$$\|\delta u_{i+1}^f(t)\| \leq \rho \|\delta u_i^f(t)\| + \alpha (\|\delta x_i(t)\| + \|z_i(t)\|) + \varepsilon. \quad (29)$$

where

$$\alpha \triangleq \max\{b_Q b_C (k_f + b_{u_d} k_B) + b_{\bar{Q}} b_C + b_{H_d} b_C \rho, b_{h_c} \rho\}$$

and

$$\varepsilon \triangleq [b_Q (b_C b_w + b_v) + b_{\bar{Q}} b_v + b_{H_d} b_v \rho].$$

By utilizing the relationship in (19), taking the  $\lambda$ -norm for (28) gives

$$\|\delta u_{i+1}^f(t)\|_\lambda \leq \rho \|\delta u_i^f(t)\|_\lambda + \alpha b_B O(|\lambda|^{-1}) \|\delta u_i^f(t)\|_\lambda + \alpha (b_{x_0} + c_0 \hat{\varepsilon}) + \varepsilon. \quad (30)$$

Referring to (20), it is clear that a sufficiently large  $|\lambda|$  can be used to ensure that

$$\rho + \alpha b_B O(|\lambda|^{-1}) \triangleq \hat{\rho} < 1. \quad (31)$$

Therefore, we have

$$b_{u^f} = \lim_{i \rightarrow \infty} \|\delta u_i^f(t)\|_\lambda = \frac{\varepsilon_0}{1 - \hat{\rho}} \triangleq b_{u^f}(b_{x_0}, b_w, b_v) \quad (32)$$

where  $\varepsilon_0 \triangleq \varepsilon + \alpha (b_{x_0} + c_0 \hat{\varepsilon})$ . From (19) and (25), (23) and (24) can be verified. It can be observed from (19) that

$$b_{xz} \triangleq \lim_{i \rightarrow \infty} (\|\delta x_i(t)\|_\lambda + \|z_i(t)\|_\lambda) \leq b_{x_0} + b_B O(|\lambda|^{-1}) b_{u^f} + c_0 \hat{\varepsilon}. \quad (33)$$

Thus, from (14), (13) and by referring to (33), we have

$$b_u \triangleq \lim_{i \rightarrow \infty} \|\delta u_i(t)\|_\lambda \leq b_{u^f} + \max\{b_{h_c}, b_{H_d} b_C\} b_{xz} + b_{H_d} b_v \triangleq b_u(b_{x_0}, b_w, b_v). \quad (34)$$

This verifies (22). Moreover, it is easy to observe that  $b_{u^f}$ ,  $b_u$ ,  $b_x$ , and  $b_e$  will all tend to zero uniformly for  $t = 0, 1, \dots, N$  as  $i \rightarrow \infty$  in the absence of uncertainty, disturbance and re-initialization error, i.e., when  $b_w, b_v, b_{x_0} \rightarrow 0$ . ■

## 5 Simulation Illustrations

To demonstrate the effectiveness of the proposed ILC scheme together with a feedback controller, a single-link manipulator model is used for simulation studies.

### 5.1 Control Problem

#### – Model for Single-link Robotic Manipulator

The dynamic equation of the single-link manipulator model in the continuous-time  $t'$  domain is

$$\ddot{\theta}(t') = \frac{1}{J} [\tau(t') + \tau_n(t')] + \frac{1}{J} \left( \frac{1}{2} m_0 + M_0 \right) g l \sin \theta(t') \quad (35)$$

where  $\theta(t')$  is the angular position of the manipulator;  $\tau(t')$  is the applied joint torque;  $\tau_n(t')$  is the exogenous disturbance torque;  $m_0$ ,  $l$  are the mass and length of the manipulator respectively,  $M_0$  is the mass of the tip load,  $g$  is the gravitational acceleration and the  $J$  is the moment of inertia w.r.t the joint, i.e.,  $J = M_0 l^2 + m_0 l^2 / 3$ .

The parameters are given as  $m_0 = 2\text{Kg}$ ,  $M_0 = 4\text{Kg}$  and  $l = 0.5\text{m}$ .  $g$  is set to  $9.8\text{m/s}^2$ . Let the sampling period  $T_s = 0.01$  sec. One can discretize the above model by using simple Euler method in the same way of [8].

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = 2x_2(t) - x_1(t) + [\tau(t) + \tau_n(t) \\ \quad + (0.5m_0 + M_0)gl \sin(x_1(t))]T_s^2/J \end{cases}$$

where discrete time  $t = 0, 1, \dots, 100$ ,  $x_1(t) = \theta(t)$ ,  $x_2(t) = \dot{\theta}(t)$ .

### – Control Task

The control objective is to track onto the desired trajectories over a fixed time interval  $[0, 1]$  sec. The tracking error should tend to zero as the control task is executed repeatedly. The desired trajectories are specified as

$$\theta_d(t) = \theta_b + (\theta_b - \theta_f)(15\hat{\tau}_0^4 - 6\hat{\tau}_0^5 - 10\hat{\tau}_0^3) \quad (36)$$

where  $\hat{\tau}_0 = tT_s/(t_f - t_0)$ . In our simulation studies, we choose  $\theta_b = 0^\circ$ ,  $\theta_f = 90^\circ$ ,  $t_0 = 0$  and  $t_f = 1$ . The initial states at each ILC repetition are all set to 0. The ILC ends when  $e_{b1} \leq 1^\circ$  where  $e_{b1} \triangleq \sup_{t \in [0, 100]} |\theta_d(t) - \theta(t)|$ .

### – Control Algorithm

*Learning Controller:* the following simple ILC updating law is used

$$u_{i+1}^f(t) = u_i(t) + Q(e_i(t+1) - e_i(t))$$

where  $Q$  is the learning gain and  $e(t) = \theta_d(t) - \theta_i(t)$ . The choose of  $Q$  is based on the condition of Theorem 4.1 where the parameters of the system are assumed to be uncertain. Note that here  $\bar{Q}$  is set to  $-Q$ .

To choose a suitable  $Q$  which make the learning convergent, we need some knowledge of the system. According to Theorem 4.1, we only need to know  $J$  and  $T_s$ . The optimal choice of  $Q$  is to make  $\rho$  be 0. This gives  $Q^* = J/T_s^2 = 10833$ . In this simulation, we use  $Q = 50$ . In this case,  $\rho$  is 0.99. This implies that we have assumed less knowledge of the system parameter  $J$ .

*Feedback Controller:* a simple P-controller

$$u_i^b(t) = K_p e_i(t)$$

is applied together with the ILC.

## 5.2 Results and Discussions

Three situations are considered. In the first two situations, there are no re-initialization errors considered. In situation 3, the random re-initialization errors are considered but with no saturater used.

- *Situation 1.* Without control saturater.

Four cases are considered for different gains  $K_p$ :

- Case 1,  $K_p = 0$ ;
- Case 2,  $K_p = 10$ ;
- Case 3,  $K_p = 20$ ;
- Case 4,  $K_p = 30$ .

For Case 1, no feedback controller is used and the tracking is by ILC only. Cases 2-4 are by ILC schemes together with feedback controllers. The results are summarized in Figure 2. It is observed that an improved ILC convergence performance can be obtained when the ILC is applied together with a feedback controller. It is interesting to note that the ILC convergence speed improves as  $K_p$  increases. However, a larger tracking error is noted in the initial iteration with a larger  $K_p$ . With this observation, we can conclude that a suitably designed feedback controller is useful for the iterative learning control.

The final converged controls are all the same for the four cases in the first situation. The control functions for various iterations are shown in Fig. 3 where the maximum amplitude of  $u_d(t)$  at the 100-th iteration is found to be around 36 Nm. It is also found that, in some of the ILC iteration processes, the control  $u_i(t)$  exceeds 36 Nm at some time instants.

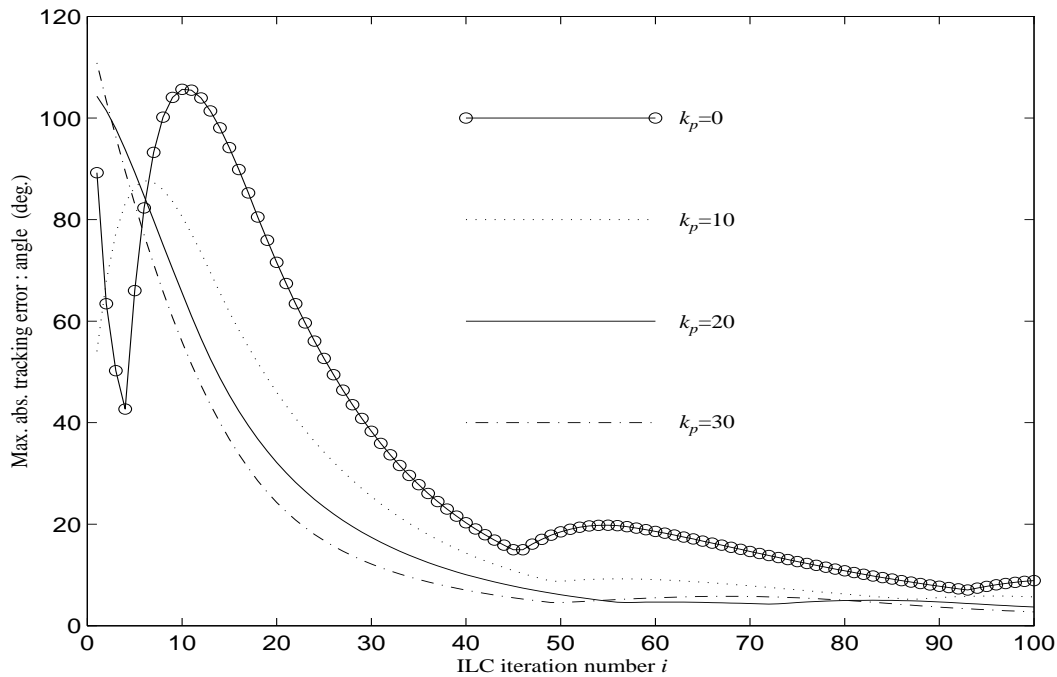


Figure 2: Convergence history comparison for schemes of ILC plus feedback controller: without input saturater

– *Situation 2. With control saturater.*

To illustrate that the ILC scheme analyzed in this paper is still effective under control saturation, we set a saturater limit  $\bar{u}^* = 36\text{Nm}$  according to A4), . Repeat the same simulation as carried out in the first situation. Similar results are obtained

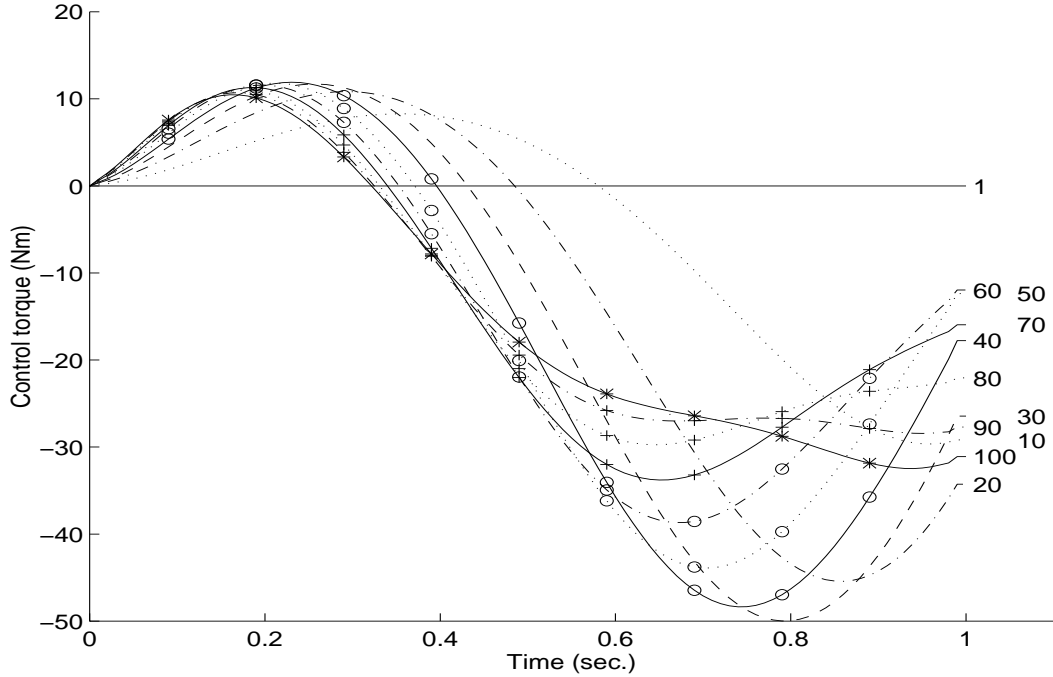


Figure 3: Control functions in various learning iterations: without input saturater ( $K_p = 0$ )

and summarized in Figure 4, which are consistent to the analysis of this paper. The control inputs at selected number of iterations are shown in Fig. 5. The control saturation can be clearly displayed. As shown in Figs. 2 and 4, it may be useful to improve learning transient by applying a control saturater at initial iterations.

- *Situation 3.* With random re-initialization errors but without control saturater .

During the learning iteration, random re-initialization errors are assumed with no control saturater. The initial state are set to  $(\text{rand}-0.5)*0.01$  rad where  $\text{rand}$  is a MATLAB function to generate uniformly distributed random variables in  $[0,1]$ . The simulation results are presented in Fig. 6 for  $K_p = 20$  and 30 respectively. The tracking error bound is a class- $K$  function of the bound of re-initialization errors  $b_{x_0}$  according to Theorem 4.1 which is illustrated by Fig. 6.

## 6 Conclusions

In this paper, a simple discrete-time iterative learning controller together with a feedback controller is proposed. Robustness convergence properties are established. Control saturation is also considered. Improved ILC performances by including feedback controller are illustrated by simulations. Further extensions under investigation include using the fractional order derivative/difference for iterative learning updating laws.

## Acknowledgments

The authors are grateful to anonymous reviewers for their constructive comments which improved the presentation of this paper.

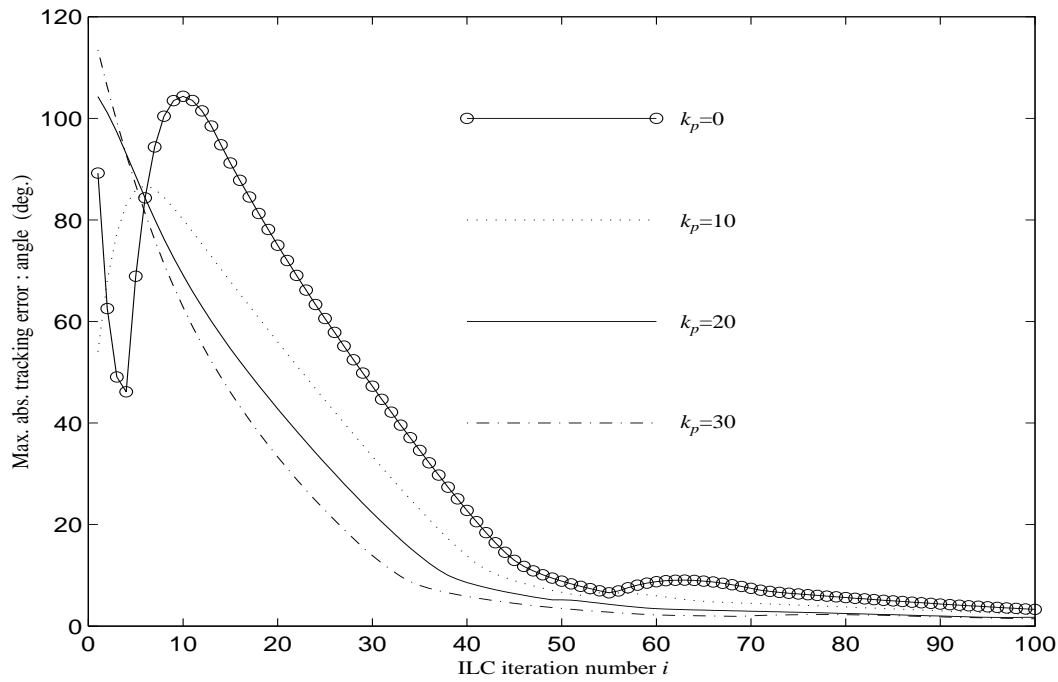


Figure 4: Convergence history comparison for schemes of ILC plus feedback controller: with input saturater

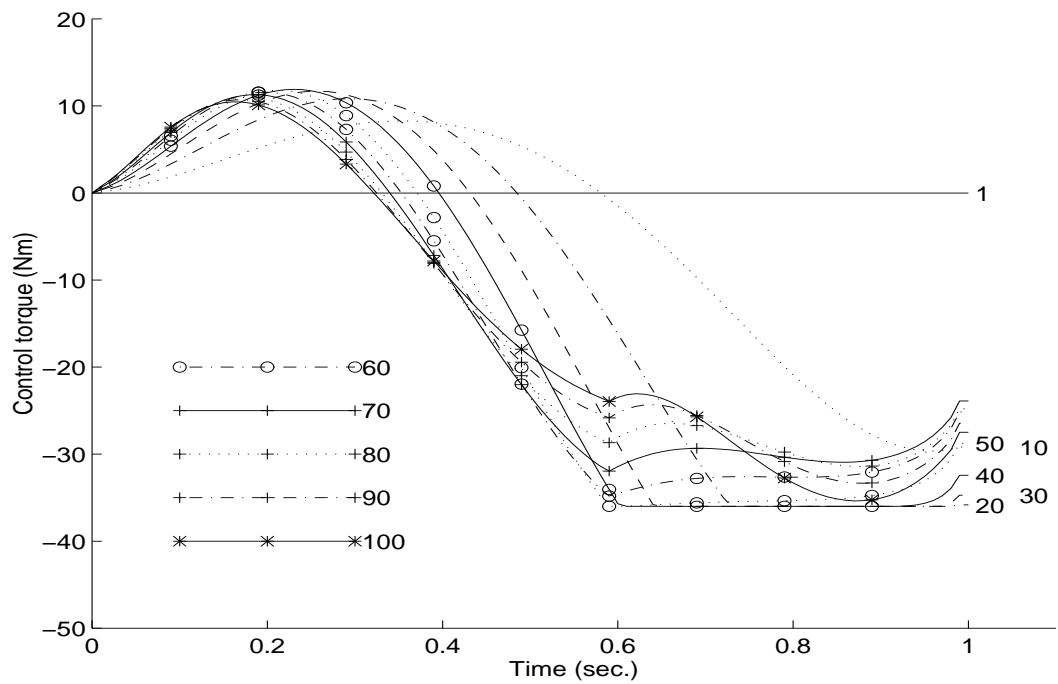


Figure 5: Control functions in various learning iterations: with an input saturater ( $K_p = 0$ )

**References**

[1] J. B. Edwards and D. H. Owens, *Analysis and Control of Multipass Processes*. Taunton, Chichester: Research Studies Press, 1982.

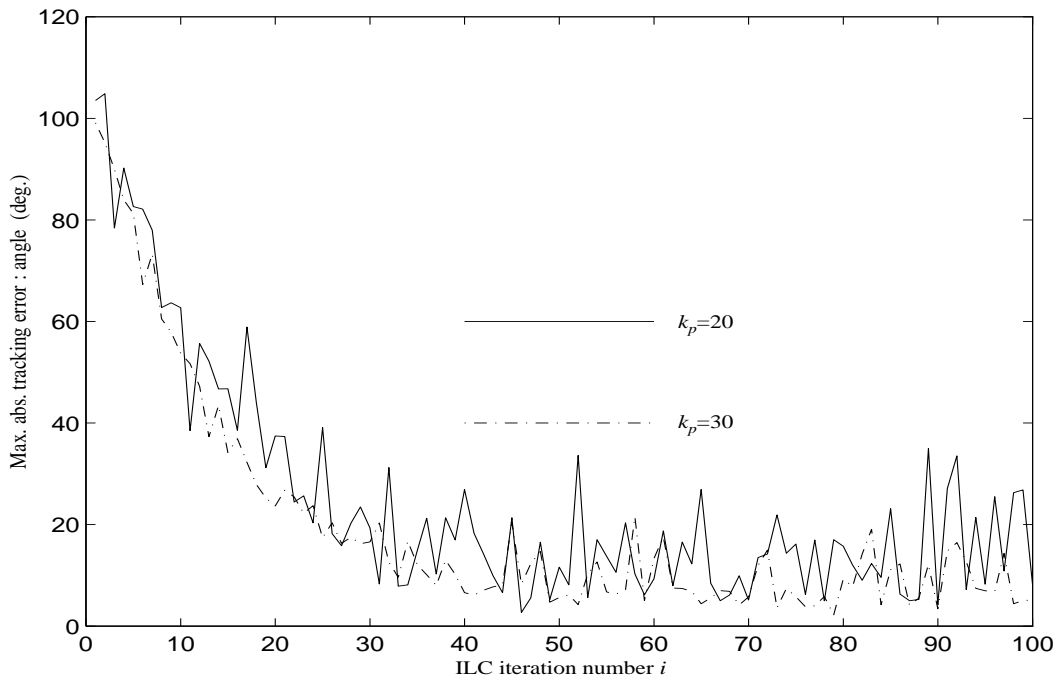


Figure 6: Convergence history comparison for schemes of ILC plus feedback controller: with random re-initialization errors

- [2] S. Arimoto, S. Kawamura, and F. Miyazaki, "Bettering operation of robots by learning," *J. of Robotic Systems*, vol. 1, no. 2, pp. 123–140, 1984.
- [3] E. Rogers and D. H. Owens, *Stability Analysis for Linear Repetitive Processes*. Berlin: Springer-Verlag, 1992.
- [4] K. L. Moore, *Iterative learning control for deterministic systems*. Advances in Industrial Control, Springer-Verlag, 1993.
- [5] Z. Geng, R. L. Carroll, and J. Xie, "Two-dimensional model algorithm analysis for a class of iterative learning control systems," *Int. J. of Control*, vol. 52, pp. 833–862, 1990.
- [6] J. E. Kurek and M. B. Zaremba, "Iterative learning control synthesis based on 2-D system theory," *IEEE Trans. of Automat. Contr.*, vol. 38, no. 1, pp. 121–125, 1993.
- [7] S. S. Saab, "A discrete-time learning control algorithm for a class of linear time-invariant systems," *IEEE Trans. of Automat. Contr.*, vol. 40, no. 6, pp. 1138–1141, 1995.
- [8] D.-H. Hwang, Z. Bien, and S.-R. Oh, "Iterative learning control method for discrete-time dynamic systems," *IEE Proc. Pt.-D*, vol. 138, no. 2, pp. 139–144, 1991.
- [9] S. K. Tso and L. Y. X. Ma, "Discrete learning control for robots: strategy, convergence and robustness," *Int. J. of Control*, vol. 57, no. 2, pp. 273–291, 1993.

- [10] T.-J. Jang, H.-S. Ahn, and C.-H. Choi, "Iterative learning control for discrete-time nonlinear systems," *Int. J. of Systems Science*, vol. 25, no. 7, pp. 1179–1189, 1994.
- [11] S. S. Saab, "Discrete-time learning control algorithm for a class of nonlinear systems," in *Proc. of American Control Conf.*, (Seattle, Washington, USA), pp. 2739–2743, June 1995.
- [12] S. Arimoto, "Robustness of learning control for robot manipulators," in *Proc. of the 1990 IEEE Int. Conf. on Robotics and Automation*, (Cincinnati, Ohio, USA), pp. 1528–1533, 1990.
- [13] T.-Y. Kuc, J. S. Lee, and K. Nam, "An iterative learning control theory for a class of nonlinear dynamic systems," *Automatica*, vol. 28, no. 6, pp. 1215–1221, 1992.
- [14] T.-J. Jang, C.-H. Choi, and H.-S. Ahn, "Iterative learning control in feedback systems," *Automatica*, vol. 31, no. 2, pp. 243–245, 1995.
- [15] Y. Chen, C. Wen, and M. Sun, "A robust high-order P-type iterative learning controller using current iteration tracking error," *Int. J. of Control*, vol. 68, pp. 331–342, Sept. 1997.
- [16] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice-Hall International, Inc., 1996.
- [17] Brian J. Driessen and Nader Sadegh, "Multi-Input Square Iterative Learning Control With Bounded Inputs," *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 124, no. 4, pp. 582–584, Dec. 2002.

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