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SUB-OPTIMUM H_2 RATIONAL APPROXIMATIONS TO FRACTIONAL ORDER LINEAR SYSTEMS

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ABSTRACT

In this paper, we propose a procedure to achieve rational approximation to arbitrary fractional order linear time invariant (FO-LTI) systems with sub-optimum H_2-norm. Through illustrations, we show that the rational approximation is simple and effective. It is also demonstrated that this sub-optimum approximation method is effective in designing integer order controllers for FO-LTI systems in general form. Useful Matlab codes are also given in the appendices.
Keywords: fractional order systems, model reduction, optimal model reduction

Introduction

Fractional order calculus, a 300-years-old topic [1–4], has been gaining increasing attention in research communities. Applying fractional-order calculus to dynamic systems control, however, is just a recent focus of interest [5–9]. We should point out references [10–13] for pioneering works and [14–16] for more recent developments. In most cases, our objective is to apply fractional order control to enhance the system control performance. For example, as in the CRONE [1] [7, 8, 17], *fractal robustness* is pursued. The desired frequency template leads to fractional transmittance [18,19] on which the CRONE controller synthesis is based. In CRONE controllers, the major ingredient is the fractional-order derivative s^r, where r is a real number and s is the Laplace transform symbol of differentiation. Another example is the PI$^\lambda$D$^\mu$ controller [6,20], an extension of PID controller. In general form, the transfer function of PI$^\lambda$D$^\mu$ is given by $K_p + T_i s^{-\lambda} + T_d s^\mu$, where λ and μ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $T_i = 0$ we obtain a PD$^\mu$ controller, etc. All these types of controllers are particular cases of the PI$^\lambda$D$^\mu$ controller. It can be expected that the PI$^\lambda$D$^\mu$ controller may enhance the systems control performance due to more tuning knobs introduced.

Actually, in theory, PI$^\lambda$D$^\mu$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of frequencies of practical interest [19,21]. Moreover, the fractional order can be a complex number as discussed in [21]. In this paper, we focus on the case where the fractional order is a real number.

For a single term s^r with r a real number, there are many approximation schemes proposed. In general, we have analog realizations [22–25] and digital realizations. The key step in digital

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[1] CRONE is a French abbreviation for "*Commande robuste d'ordre non-entier*" (which means non-integer order robust control).

implementation of an FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods [21], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator [26–29], continuous fractional expansion (CFE) of the Tustin operator [27–31], and numerical integration based method [26, 30, 32]. However, as pointed out in [33–35], the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [30] which yields the so-called Al-Alaoui operator [33]. These discretization methods for s^r are in IIR form. Recently, there are some reported methods to directly obtain the digital fractional order differentiators in FIR (finite impulse response) form [36, 37]. However, using an FIR filter to approximate s^r may be less efficient due to very high order of the FIR filter. So, discretizing fractional differentiators in IIR forms is perferred [30–32, 38].

In this paper, we consider the general fractional order LTI systems (FO-LTI) with noncommensurate fractional orders as follows:

$$G(s) = \frac{b_m s^{\gamma_m} + b_{m-1} s^{\gamma_{m-1}} + \cdots + b_1 s^{\gamma_1} + b_0}{a_n s^{\eta_n} + a_{n-1} s^{\eta_{n-1}} + \cdots + a_1 s^{\eta_1} + a_0}. \tag{1}$$

Using the aforementioned approximation schemes for a single s^r and then for the general FO-LTI system (1) could be very tedious, leading to a very high order. In this paper, we propose to use a numerical algorithm to achieve a good approximation of the overall transfer function (1) using finite integer-order rational transfer function and illustrate how to use the approximated integer-order model for controller design. In examples 1 and 2, approximation to a fractional order transfer function is given and the fitting results are illustrated. In example 3, a fractional order plant is approximated using the algorithm proposed in the paper, by a FOPD (first-order plus delay) model, and using an existing PID tuning formula, an integer order PID can be designed with very good performance.

Useful in design of fractional order systems, using the integer order methods

Rational Approximations to Fractional Integrators and Differentiators: Outstaloup's Method

For comparison purpose, here we present the Oustaloup algorithm [18, 19, 39]. Assuming that the frequency range to fit is selected as (ω_b, ω_h), the transfer function of a continuous filter can be constructed such that

$$G_f(s) = K \prod_{k=-N}^{N} \frac{s+\omega_k'}{s+\omega_k} \tag{2}$$

where the zeros, poles and the gain can be evaluated from

$$\begin{aligned} \omega_k' &= \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}} \\ \omega_k &= \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}} \\ K &= \left(\frac{\omega_h}{\omega_b}\right)^{-\frac{\gamma}{2}} \prod_{k=-N}^{N} \frac{\omega_k}{\omega_k'}. \end{aligned} \tag{3}$$

An implementation in Matlab is given in Appendix 1. Substituting γ_i and η_i in (1) with $G_{f,\gamma_i}(s)$ and $G_{f,\eta_i}(s)$ respectively, the original fractional order model $G(s)$ can be approximated by a rational function $\widehat{G}(s)$. It should noted that the order of the resulted $\widehat{G}(s)$ is usually very high. Thus, there is a need to approximate the original model by reduced order ones using the optimal reduction techniques.

A Numerical Algorithm for Sub-optimal Rational Approximations

In this section, we are interested in finding an approximate integer order model with a low order, possibly with a time delay in the following form:

$$G_{r/m,\tau}(s) = \frac{\beta_1 s^r + \ldots + \beta_r s + \beta_{r+1}}{s^m + \alpha_1 s^{m-1} + \ldots + \alpha_{m-1} s + \alpha_m} e^{-\tau s}. \tag{4}$$

An objective function for sub-optimal model reduction can be defined as

$$J = \min_{\theta} \left\| \widehat{G}(s) - G_{r/m,\tau}(s) \right\|_2 \tag{5}$$

where θ is the set of parameters to be optimized such that

$$\theta = [\beta_1, \ldots, \beta_r, \alpha_1, \ldots, \alpha_m, \tau]. \tag{6}$$

For an easy evaluation of the criterion J, the delayed term in the reduced order model can be further approximated by a rational function using the Padé approximation technique [40]. Thus, the revised criterion can then be defined by

$$J = \min_{\theta} \left\| \widehat{G}(s) - \widehat{G}_{r/m}(s) \right\|_2. \tag{7}$$

The above optimization problem can be solved using the following procedure [40]:

1. Select an initial reduced model $\widehat{G}^0_{r/m}(s)$.
2. Obtain an error function $\left\| \widehat{G}(s) - \widehat{G}^*_{r/m}(s) \right\|_2$.
3. Use an optimization algorithm (for instance, Powell's algorithm [41]) to iterate one step for a better estimated model $\widehat{G}^1_{r/m}(s)$.
4. Set $\widehat{G}^0_{r/m}(s) \leftarrow \widehat{G}^1_{r/m}(s)$, go to step 2 until an optimal reduced model $\widehat{G}^*_{r/m}(s)$ is obtained.
5. Extract the delay from $\widehat{G}^*_{r/m}(s)$, if any.

We call the above procedure sub-optimal since the Oustaloup's method is used for each single term s^r in (1).

Illustrative Examples

Example 1

Consider the non-commensurate FO-LTI system

$$G(s) = \frac{5}{s^{2.3} + 1.3s^{0.9} + 1.25}.$$

Using the following Matlab scripts,

```
w1=1e-3; w2=1e3; N=2;
g1=ousta_fod(0.3,N,w1,w2);
g2=ousta_fod(0.9,N,w1,w2);
s=tf('s');
G=5/(s^2*g1+1.3*g2+1.25);
     % original model
G1=opt_app(G,1,2,0); % order 1/2
G2=opt_app(G,2,3,0); % order 2/3
G3=opt_app(G,3,4,0); % order 3/4
G4=opt_app(G,4,5,0); % order 4/5
step(G,G1,G2,G3,G4)
```

the four step responses can be obtained as compared in Fig. 1.

The obtained optimum approximated results are listed in the following:

$$G_1(s) = \frac{-2.045s + 7.654}{s^2 + 1.159s + 1.917}$$

$$G_2(s) = \frac{-0.5414s^2 + 4.061s + 2.945}{s^3 + 0.9677s^2 + 1.989s + 0.7378}$$

$$G_3(s) = \frac{-0.2592s^3 + 3.365s^2 + 4.9s + 0.3911}{s^4 + 1.264s^3 + 2.25s^2 + 1.379s + 0.09797}$$

$$G_4(s) = \frac{1.303s^4 + 1.902s^3 + 11.15s^2 + 4.71s + 0.1898}{s^5 + 2.496s^4 + 3.485s^3 + 4.192s^2 + 1.255s + 0.04755}$$

Clearly, the low order approximation using the method and codes of this paper is effective. Note that the original model G after using the Oustaloup's method is

$$G(s) = \frac{\begin{array}{c} 5s^{10} + 6677s^9 + 2.191e06s^8 + 1.505e08s^7 + \\ 2.936e09s^6 + 1.257e010s^5 + 1.541e010s^4 + \\ 4.144e09s^3 + 3.168e08s^2 + 5.065e06s + 1.991e04 \end{array}}{\begin{array}{c} 7.943s^{12} + 8791s^{11} + 1.731e06s^{10} + 8.766e07s^9 + \\ 1.046e09s^8 + 3.82e09s^7 + 6.099e09s^6 + 7.743e09s^5 \\ +5.197e09s^4 + 1.15e09s^3 + 8.144e07s^2 + 1.278e06s + 4987 \end{array}}.$$

Example 2

Consider the following non-commensurate FO-LTI system:

$$G(s) = \frac{5s^{0.6} + 2}{s^{3.3} + 3.1s^{2.6} + 2.89s^{1.9} + 2.5s^{1.4} + 1.2}.$$

Using the following Matlab scripts,

```
N=2; w1=1e-3; w2=1e3;
g1=ousta_fod(0.3,N,w1,w2);
g2=ousta_fod(0.6,N,w1,w2);
g3=ousta_fod(0.9,N,w1,w2);
g4=ousta_fod(0.4,N,w1,w2);
s=tf('s');
G=(5*g2+2)/(s^3*g1+3.1*s^2*g2 ...
  +2.89*s*g3+2.5*s*g4+1.2);
  % original model
G2=opt_app(G,2,3,0); % order 2/3
G3=opt_app(G,3,4,0); % order 3/4
G4=opt_app(G,4,5,0); % order 4/5
step(G,G2,G3,G4),xlim([0,60])
```

the step responses can be compared in Fig. 1 and the obtained optimum approximated results are listed in the following:

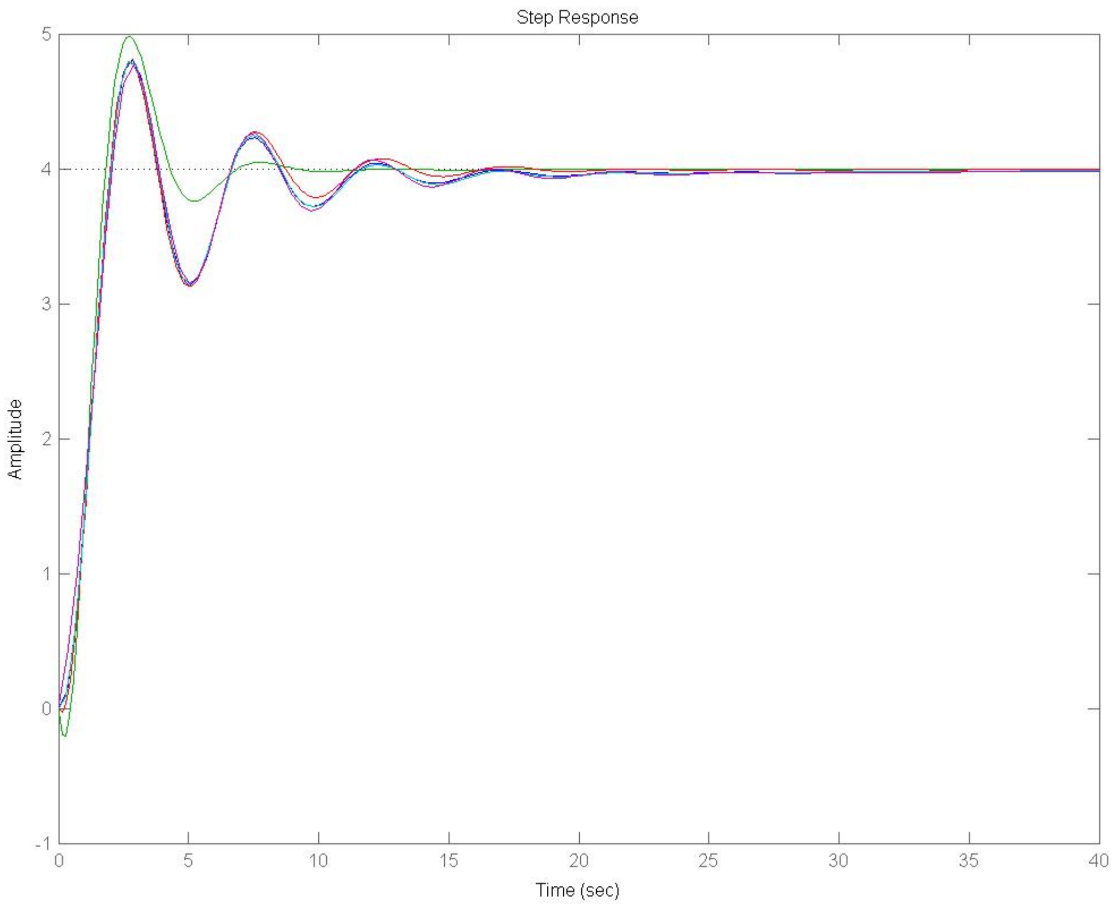

Figure 1. Step responses of rational approximations with different orders for Example-1

$$G_2(s) = \frac{0.41056s^2 + 0.75579s + 0.037971}{s^3 + 0.24604s^2 + 0.22176s + 0.021915}$$

$$G_3(s) = \frac{-4.4627s^3 + 5.6139s^2 + 4.3354s + 0.15330}{s^4 + 7.4462s^3 + 1.7171s^2 + 1.5083s + 0.088476}$$

$$G_4(s) = \frac{1.7768s^4 + 2.2291s^3 + 10.911s^2 + 1.2169s + 0.010249}{s^5 + 11.347s^4 + 4.8219s^3 + 2.8448s^2 + 0.59199s + 0.0059152}$$

Clearly, the low order approximation using the method and codes of this paper is effective. Note that the original model G after using the Oustaloup's method is

$$G(s) = \frac{\begin{array}{c} 317.5s^{25} + 8.05e005s^{24} + 7.916e008s^{23} + 3.867e011s^{22} + \\ 1.001e014s^{21} + 1.385e016s^{20} + 1.061e018s^{19} + 4.664e019s^{18} + \\ 1.197e021s^{17} + 1.778e022s^{16} + 1.5e023s^{15} + 7.242e023s^{14} + \\ 2.052e024s^{13} + 3.462e024s^{12} + 3.459e024s^{11} + 2.009e024s^{10} + \\ 6.724e023s^{9} + 1.329e023s^{8} + 1.579e022s^{7} + 1.12e021s^{6} + 4.592e019s^{5} + \\ 1.037e018s^{4} + 1.314e016s^{3} + 9.315e013s^{2} + 3.456e011s + 5.223e008 \end{array}}{\begin{array}{c} 7.943s^{28} + 2.245e004s^{27} + 2.512e007s^{26} + 1.427e010s^{25} + \\ 4.392e012s^{24} + 7.384e014s^{23} + 6.896e016s^{22} + 3.736e018s^{21} + \\ 1.208e020s^{20} + 2.343e021s^{19} + 2.716e022s^{18} + 1.896e023s^{17} + \\ 8.211e023s^{16} + 2.268e024s^{15} + 4.076e024s^{14} + 4.834e024s^{13} + \\ 3.845e024s^{12} + 2.134e024s^{11} + 8.772e023s^{10} + 2.574e023s^{9} + \\ 5.057e022s^{8} + 6.342e021s^{7} + 4.868e020s^{6} + 2.16e019s^{5} + \\ 5.176e017s^{4} + 6.863e015s^{3} + 5.055e013s^{2} + 1.938e011s + 3.014e008 \end{array}}.$$

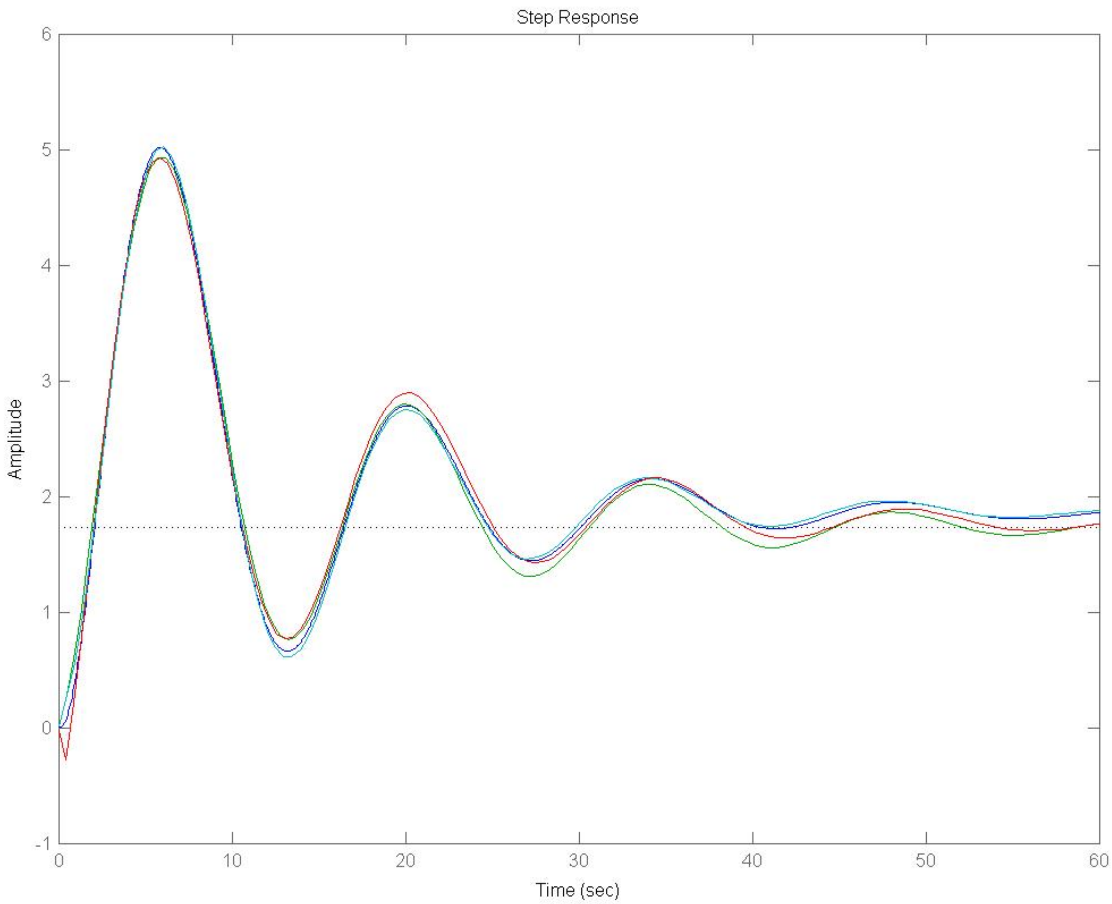

Figure 2. Step responses of rational approximations with different orders for Example-2

Example 3, sub-optimum model reduction for integer-order PID controller design

Let us consider the following FO-LTI system:

$$G(s) = \frac{1}{s^{2.3} + 3.2s^{1.4} + 2.4s^{0.9} + 1}.$$

Let us first approximate it with Oustaloup's method and then fit it with a fixed model structure known as FOPD (first-order plus delay). The following Matlab scripts

```
N=2; w1=1e-3; w2=1e3;
g1=ousta_fod(0.3,N,w1,w2);
g2=ousta_fod(0.4,N,w1,w2);
g3=ousta_fod(0.9,N,w1,w2);
s=tf('s');
G=1/(s^2*g1+3.2*s*g2+2.4*g3+1);
    % original model
G2=opt_app(G,0,1,1);
step(G,G2)
```

can perform this task and the obtained optimal FOPD model is given as follows:

$$G_r(s) = \frac{0.9951}{3.5014s + 1}e^{-0.634s}.$$

The comparison of the open-loop step response is shown in Fig. 3. It can be observed that the approximation is effective.

Designing a suitable feedback controller for the original FO-LTI system G can be a formidable task. Now, let us consider designing an integer order PID controller for the optimally reduced model $G_r(s)$ and let us see if the designed controller still works for the original system.

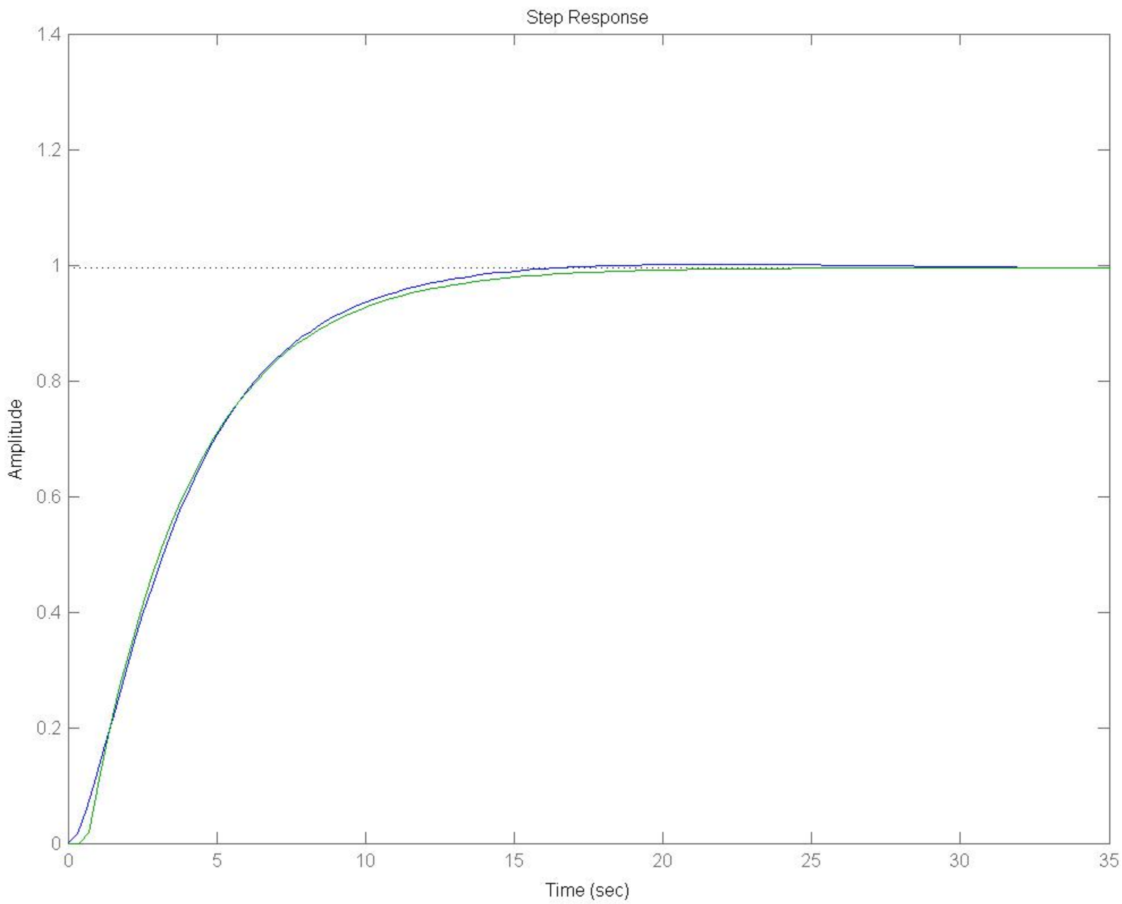

Figure 3. Step response comparison of the optimum FOPD and the original model

The integer order PID controller to be designed is in the following form:

$$G_c(s) = K_p\left(1+\frac{1}{T_i s}+\frac{T_d s}{T_d/Ns+1}\right). \tag{8}$$

Here, we simply apply an optimum ITAE criterion-based PID tuning formula [42]:

$$K_p = \frac{(0.7303+0.5307T/L)(T+0.5L)}{K(T+L)}, \tag{9}$$

$$T_i = T+0.5L,\ \ T_d = \frac{0.5LT}{T+0.5L}. \tag{10}$$

Based on this tuning algorithm, a PID controller can be designed for $G_r(s)$ as follows:

```
  L=0.63; T=3.5014; K=0.9951; N=10;
  Ti=T+0.5*L;
  Kp=(0.7303+0.5307*T/L)*Ti/(K*(T+L));
  Td=(0.5*L*T)/(T+0.5*L); [Kp,Ti,Td]
  Gc=Kp*(1+1/Ti/s+Td*s/(Td/N*s+1))
  Transfer function:
1.086 s^2 + 3.442 s + 0.8951
----------------------------
     0.0289 s^2 + s
Kp =
    3.4160
Ti =
    3.8164
Td =
    0.2890
```

Finally, the step response of the original FO-LTI with the above designed PID controller is shown in Fig. 4. A satisfactory performance can be clearly observed. Therefore, we believe, the method presented in this paper can be used for integer order controller design for general FO-LTI systems. The design can be simply based on an approximated model with possible "*artificial delay*". It is interesting to observe that the simple PID control achieves reasonable performance for fractional order systems although by intuition that a better controller would also be fractional order. Considering the implementation cost for fractional order controllers, integer order controllers, if well designed, for fractional order systems should also be practically attractive.

Conclusion

In this paper, we presented a procedure to achieve rational approximation to arbitrary fractional order linear time invariant (FO-LTI) systems with sub-optimum H_2-norm. Relevant Matlab codes useful for practical applications are also given in four appendices. Through illustrations, we show that the rational approximation is simple and effective. It is also demonstrated that this sub-optimum approximation method is effective in designing integer order controllers for FO-LTI systems in general form.

Appendix 1: `ousta_fod.m`

Outstaloup's rational approximation.

```
function G=ousta_fod(r,N,w_L,w_H)
% r: fractional order, r \in (-1,1)
% N: order
% w_L: low frequency limit
% w_H: upper frequency limit
% G: the returned LTI object to approx. s^r
mu=w_H/w_L; k=-N:N;
w_kp=(mu).^((k+N+0.5-0.5*r)/(2*N+1) )*w_L;
w_k=(mu).^((k+N+0.5+0.5*r)/(2*N+1) )*w_L;
K=(mu)^(-r/2)*prod(w_k./w_kp);
G=tf(zpk(-w_kp',-w_k',K));
```

Appendix 2: `opt_app.m`

```
function G_r=opt_app(G_Sys,nn,nd,key,G0)
%OPT_APP finds the optimally reduced model.
%
%G_r=opt_app(G_Sys,nn,nd,key,G0)
% G_Sys is the original model.
% nn, dd: the expected orders of num & dem.
% key is the id to show whether delay
%   is allowed in the reduced order model.
% G0 is the initial model, if needed.
% Gr is the reduced order model.

%Designed by Professor Dingyu Xue
%Northeastern Univ., Shenyang 110004, China
%1st drafted'98, revised for LFC book, 2002
GS=tf(G_Sys);
num=GS.num{1}; den=GS.den{1};
Td=totaldelay(GS);
GS.ioDelay=0;
GS.InputDelay=0;GS.OutputDelay=0;
if nargin<5,
   n0=[1,1];
   for i=1:nd-2, n0=conv(n0,[1,1]); end
   G0=tf(n0,conv([1,1],n0));
end
beta=G0.num{1}(nd+1-nn:nd+1);
alph=G0.den{1}; Tau=1.5*Td;
x=[beta(1:nn),alph(2:nd+1)];
if abs(Tau)<1e-5, Tau=0.5; end
if key==1, x=[x,Tau]; end
dc=dcgain(GS);
y=opt_fun(x,GS,key,nn,nd,dc);
x=fminsearch('opt_fun',x,[],GS,key,nn,...
    nd,dc);
alph=[1,x(nn+1:nn+nd)]; beta=x(1:nn+1);
if key==0, Td=0; end
beta(nn+1)=alph(end)*dc;
if key==1, Tau=x(end)+Td;
else, Tau=0; end
G_r=tf(beta,alph,'ioDelay',Tau);
```

Appendix 3: `opt_fun.m`

```
function y=opt_fun(x,G,key,nn,nd,dc)
%OPT_FUN is called by OPT_APP.
%not recommended to call manually.

%Designed by Professor Dingyu Xue
%Northeastern Univ., Shenyang 110004, China
%1st drafted'90, revised for LFC book, 2002
ff0=1e10; alph=[1,x(nn+1:nn+nd)];
beta=x(1:nn+1); beta(end)=alph(end)*dc;
g=tf(beta,alph);
if key==1,
   tau=x(end); if tau<=0, tau=eps; end
   [nP,dP]=pade(tau,3); gP=tf(nP,dP);
else,
   gP=1;
end
G_e=G-g*gP;
G_e.num{1}=[0,G_e.num{1}(1:end-1)];
[y,ierr]=geth2(G_e);
if ierr==1, y=10*ff0; else, ff0=y; end
```

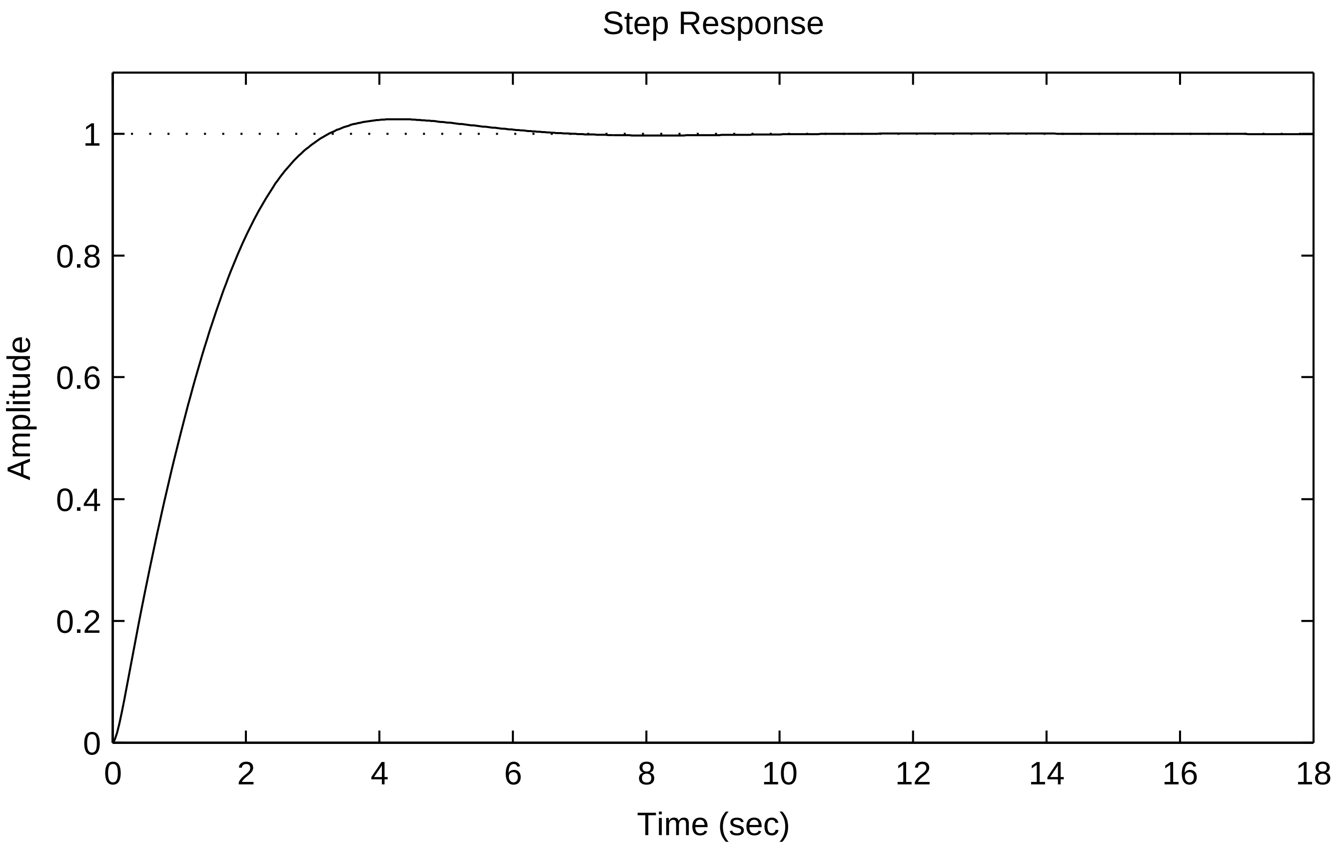

Figure 4. Step response of fractional order plant model using integer order PID controller

Appendix 4: `get2h.m`

```
function [v,ierr]=geth2(G)
%GETH2 computes the H2 norm of a system
% using K. J. Astrom's recursive method.
%
%[v,ierr]=GETH2(G)
% G is the system model,
% v is the H2 norm.
% If ierr=1, then the system is unstable,
%                  and v is useless.

%  See Also: NORMH2, COVAR

%Designed by Professor Dingyu Xue
%Northeastern Univ., Shenyang 110004, China
%1st drafted'90, revised for LFC book, 2002
G=tf(G); num=G.num{1}; den=G.den{1};
ierr=0; v=0; n=length(den);
if abs(num(1))>eps
   disp('System not strictly proper');
   ierr=1; return
else, a1=den; b1=num(2:length(num)); end
for k=1:n-1
  if (a1(k+1)<=eps), ierr=1; return
  else,
     aa=a1(k)/a1(k+1); bb=b1(k)/a1(k+1);
     v=v+bb*bb/aa; k1=k+2;
     for i=k1:2:n-1
        a1(i)=a1(i)-aa*a1(i+1);
        b1(i)=b1(i)-bb*a1(i+1);
end, end, end
v=sqrt(0.5*v);
```

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