**Proceedings of IDETC/CIE 2005**
**ASME 2005 International Design Engineering Technical Conferences**
**& Computers and Information in Engineering Conference**
**September 24-28, 2005, Long Beach, California USA**

**DETC2005-85299**

# ROBUSTNESS OF BOUNDARY CONTROL OF FRACTIONAL WAVE EQUATIONS WITH DELAYED BOUNDARY MEASUREMENT USING FRACTIONAL ORDER CONTROLLER AND THE SMITH PREDICTOR

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## ABSTRACT

In this paper, we analyze the robustness of the fractional wave equation with a fractional order boundary controller subject to delayed boundary measurement. Conditions are given to guarantee stability when the delay is small. For large delays, the Smith predictor is applied to solve the instability problem and the scheme is proved to be robust against a small difference between the assumed delay and the actual delay. The analysis shows that fractional order controllers are better than integer order controllers in the robustness against delays in the boundary measurement.

**Keywords**: fractional wave equation, fractional order controller, Smith predictor

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## 1 Introduction

In recent years, boundary control of flexible systems has become an active research area, due to the increasing demand on high precision control of many mechanical systems, such as spacecraft with flexible attachment or robots with flexible links, which are governed by PDEs (partial differential equations) rather than ODEs (ordinary differential equations) [1–9]. In this research area, the robustness of controllers against delays is an important topic and has been studied by many researchers [10–15], due to the fact that delays are unavoidable in practical engineering.

Fractional diffusion and wave equations are obtained from the classical diffusion and wave equations by replacing the first and second order time derivative term by a fractional derivative of an order satisfying $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$, respectively. Since many of the universal phenomenons can be modelled accurately using the fractional diffusion and wave equations (see [16]), there has been a growing interest in investigating the solutions and properties of these evolution equations. Compared with the publications on control of integer order PDEs, results on control of fractional wave equations are relatively few [17, 18]. To the best of the authors' knowledge, there is still no publication on robust stabilization of fractional wave equations subject to delayed boundary measurement.

In this paper, we will investigate two robust stabilization problems of the fractional wave equations subject to delayed boundary measurement. First, under what conditions a very small delay in boundary measurement will not cause instability problems. Second, how to stabilize the system when the delay is large enough and makes the system unstable.

The paper is organized as follows. In Sec. 2, the mathematical formulation is given. The robustness of boundary stabilization of fractional wave equation subject to a small delay in boundary measure is analyzed in Sec. 3. Section 4 investigates the large delay case and the corresponding compensation scheme. Finally, Section 5 concludes this paper.

## 2 Problem Formulation

We consider a cable made with special smart materials governed by the fractional wave equation, fixed at one end, and stabilized by a boundary controller at the other end. Omitting the mass of the cable, the system can be represented by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \alpha \le 2, \quad x \in [0,1], \quad t \ge 0 \tag{1}$$

$$u(0,t) = 0, \tag{2}$$
$$u_x(1,t) = f(t), \tag{3}$$
$$u(x,0) = u_0(x), \tag{4}$$
$$u_t(x,0) = v_0(x), \tag{5}$$

where $u(x,t)$ is the displacement of the cable at $x \in [0,1]$ and $t \ge 0$, $f(t)$ is the boundary control force at the free end of the cable, $u_0(x)$ and $v_0(x)$ are the initial conditions of displacement and velocity, respectively.

The control objective is to stabilize $u(x,t)$, given the initial conditions (4) and (5).

We adopt the following Caputo definition for fractional derivative of order $\alpha$ of any function $f(t)$, because the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations [19, 20]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \tag{6}$$

where $n$ is an integer satisfying $n-1 < \alpha \le n$ and $\Gamma$ is the Euler's `Gamma` function.

In this paper, we study the robustness of the controllers in the following format:

$$f(t) = -k \frac{\mathrm{d}^\mu u(1,t)}{\mathrm{d}t^\mu}, \quad 0 < \mu \le 1 \tag{7}$$

where $k$ is the controller gain, $\mu$ is the order of fractional derivative of the displacement at the free end of the cable.

Based on the definition (6), the Laplace transform of the fractional derivative is [19, 20]:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+) s^{\alpha-1-k} \tag{8}$$

In the following, the transfer function from the boundary controller $f(t)$ to the tip end displacement will be derived for later use.

Assuming zero initial conditions of $u(x,0)$ and $u_t(x,0)$, take the Laplace transform of (1), (2), and (3) with respect to $t$, making use of (8), the original PDE of $u(x,t)$ with initial and boundary conditions can be transformed into the following ODE of $U(x,s)$ with boundary conditions.

$$\frac{\mathrm{d}^2 U(x,s)}{\mathrm{d}x^2} - s^\alpha U(x,s) = 0, \tag{9}$$

$$U(0,s) = 0, \tag{10}$$

$$U_x(1,s) = F(s), \tag{11}$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$ and $F(s)$ is the Laplace transform of $f(t)$.

Solving the ODE (9), we have the following solution of $U(x,s)$ with two arbitrary constants $C_1$ and $C_2$ ($s$ can be treated as a constant in this step).

$$U(x,s) = C_1 e^{xs^{\frac{\alpha}{2}}} + C_2 e^{-xs^{\frac{\alpha}{2}}}. \tag{12}$$

Substitute (12) into (10) and (11), we have the following two equations.

$$C_1 + C_2 = 0, \tag{13}$$

$$s^{\frac{\alpha}{2}}(C_1 e^{s^{\frac{\alpha}{2}}} - C_2 e^{-s^{\frac{\alpha}{2}}}) = F(s). \tag{14}$$

Solving (13) and (14) simultaneously, we can obtain the exact value of $C_1$ and $C_2$

$$C_1 = -C_2 = \frac{F(s)e^{s^{\frac{\alpha}{2}}}}{s^{\frac{\alpha}{2}}(e^{2s^{\frac{\alpha}{2}}}+1)}. \tag{15}$$

Now we have obtained the solution of $U(x,s)$. Substituting $x=1$ into $U(x,s)$ and divide $U(x,s)$ by $F(s)$, we obtain the following transfer function of the fractional wave equation $P(s)$:

$$P(s) = \frac{U(1,s)}{F(s)} = \frac{1-e^{-2s^{\frac{\alpha}{2}}}}{s^{\frac{\alpha}{2}}\left(1+e^{-2s^{\frac{\alpha}{2}}}\right)}. \tag{16}$$

## 3 Robustness of Boundary Stabilization Subject to A Small Delay in Boundary Measurement

We consider the presence of a very small time delay $\theta$ in boundary measurement, shown as follows

$$f(t) = -ku_t^{(\mu)}(1, t-\theta), \tag{17}$$

where $\theta$ is the time delay.

The situation is also illustrated in Fig.1, where $P(s)$ is the transfer function of the plant and $C(s)$ is the Laplace transform of the controller. In our case, $P(s)$ is (16) and $C(s)$ is

$$C(s) = ks^{\mu} \tag{18}$$

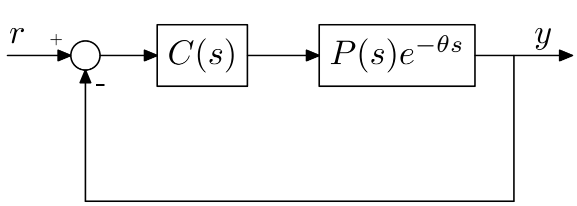


Figure 1. A feedback control system with a time delay

In [10–13], it was shown that an arbitrarily small delay in boundary measurement causes the instability problem in boundary control of wave equations using integer order controllers $f(t) = -ku_t(1,t)$. Does this problem exist in boundary control of the fractional wave equation? Since fractional order controllers are chosen in this paper, will this additional tuning knob bring us any benefits of robustness against the small delay? To answer these questions, we will first introduce a theorem presented in [12, 13].

**Theorem 1.** *Let $H(s)$ be the open-loop transfer function as illustrated in Fig. 2 and $\mathfrak{D}_H$ the set of all its poles. Define two closed-loop transfer functions $G_0(s)$ and $G_\varepsilon(s)$ as*

$$G_0(s) = \frac{H(s)}{1+H(s)},$$

*and*

$$G_\varepsilon(s) = \frac{H(s)}{1+e^{-\varepsilon s}H(s)}.$$

*Define again*

$$\mathbb{C}_0 = \{s \in \mathbb{C} | \Re(s) > 0\},$$

*and*

$$\gamma(H(s)) = \limsup_{|s|\to\infty, s\in\mathbb{C}_0\setminus\mathfrak{D}_H} |H(s)|.$$

*Suppose $G_0$ is $L^2$-stable. If $\gamma(\mathrm{H}) < 1$, then there exists $\varepsilon^*$ such that $\mathrm{G}_\varepsilon$ is $L^2$-stable for all $\varepsilon \in (0, \varepsilon^*)$.*

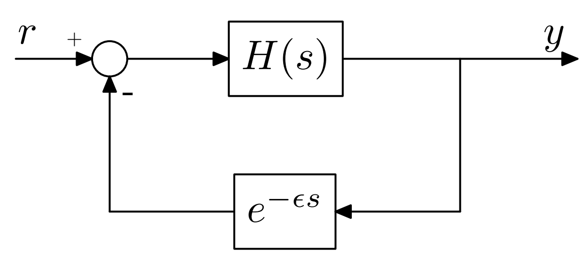


Figure 2. Feedback system with delay

The underlying idea of the above theorem is that the robustness of the closed-loop transfer function $G_0(s)$ against a small unknown delay can be determined by studying the open-loop transfer function $H(s)$. Notice that $H(s) = C(s)P(s)$ in our case.

CLAIM:

If the derivative order $\mu$ of controller (7) and the fractional order $\alpha$ in the fractional wave equation (1) satisfy

$$\mu < \frac{\alpha}{2}, \tag{19}$$

then the system is stable for a small enough delay $\theta$ in boundary measurement.

*Proof*:

For $s \in \mathbb{C}_0$,

$$\begin{aligned} |H(s)| &= |C(s)P(s)| \\ &= \left| \frac{ks^{\mu}(1-e^{-2s^{\frac{\alpha}{2}}})}{s^{\frac{\alpha}{2}}\left(1+e^{-2s^{\frac{\alpha}{2}}}\right)} \right| \\ &= \left| \frac{k(1-e^{-2s^{\frac{\alpha}{2}}})}{s^{(\frac{\alpha}{2}-\mu)}\left(1+e^{-2s^{\frac{\alpha}{2}}}\right)} \right| \\ &\leq \frac{k|1-e^{-2s^{\frac{\alpha}{2}}}|}{|s^{(\frac{\alpha}{2}-\mu)}||1+e^{-2s^{\frac{\alpha}{2}}}|} \end{aligned} \tag{20}$$

Since $\frac{\alpha}{2} > \mu$, $|s^{(\frac{\alpha}{2}-\mu)}| \to \infty \quad$ for $|s| \to \infty$.

Since $\frac{1}{2} < \frac{\alpha}{2} < 1$, for $|s|$ large enough, $|1-e^{-2s^{\frac{\alpha}{2}}}|$ is bounded and $|1-e^{-2s^{\frac{\alpha}{2}}}| > \eta > 0$, where $\eta$ is a positive number.

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| = 0 < 1.$$

Following the above proof, it can be easily proved that an integer order controller $f(t) = -ku_t(1,t)$ is not robust against an arbitrarily small delay.

## 4 Compensation of Large Delays in Boundary Measurement Using the Smith Predictor

In the last section, it is shown that an fractional order controller is robust against a small delay under the condition (19). In this section, we investigate the problem that what if the delay is large and makes the system unstable? We will apply the Smith predictor to solve this problem.

### 4.1 A Brief Introduction to The Smith Predictor

The Smith predictor was proposed by Smith in [21] and is probably the most famous method for control of systems with time delays [22, 23]. Consider a typical feedback control system with a time delay in Fig. 1, where $C(s)$ is the controller; $P(s)e^{-\theta s}$ is the plant with a time delay $\theta$.

With the presence of the time delay, the transfer function of the closed-loop system relating the output $y(s)$ to the reference $r(s)$ becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1+C(s)P(s)e^{-\theta s}}. \tag{21}$$

Obviously, the time delay $\theta$ directly changes the closed-loop poles. Usually, the time delay reduces the stability margin of the control system, or more seriously, destabilizes the system.

The classical configuration of a system containing a Smith predictor is depicted in Fig. 3, where $\hat{P}_0(s)$ is the assumed model of $P_0(s)$ and $\hat{\theta}$ is the assumed delay. The block $C(s)$ combined with the block $\hat{P}(s) - \hat{P}(s)e^{-\hat{\theta}s}$ is called "the Smith predictor". If we assume the perfect model matching, *i.e.*, $\hat{P}_0(s) = P_0(s)$ and $\theta = \hat{\theta}$, the closed-loop transfer function becomes

$$\frac{y(s)}{r(s)} = \frac{C(s)P(s)e^{-\theta s}}{1+C(s)P(s)}. \tag{22}$$

Now, it is clear what the underlying idea of the Smith predictor is. With the perfect model matching, the time delay can be removed from the denominator of the transfer function, making the closed-loop stability irrelevant to the time delay.

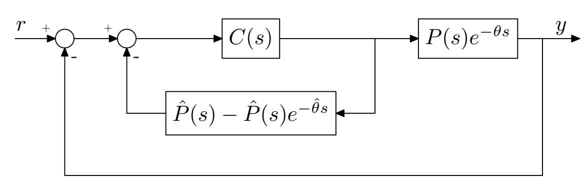


Figure 3. The Smith predictor

Based on the controller (18) as $C(s)$,we have the following expression of the boundary controller (the Smith predictor), denoted as $C_{sp}(s)$:

$$C_{sp}(s) = \frac{ks^{\mu}}{1+ks^{\mu}P(s)(1-e^{-\hat{\theta}s})} \tag{23}$$

### 4.2 Robustness Analysis of the Smith predictor

In Sec. 4.1, it is shown that if the assumed delay is equal to the actual delay, the Smith predictor removes the delay term completely from the denominator of the closed-loop. However, the actual delay is not exactly known. In this section, we will investigate what if an unknown small difference $\varepsilon$ between the assumed delay and the actual delay is introduced to the system, as shown in Fig. 4.

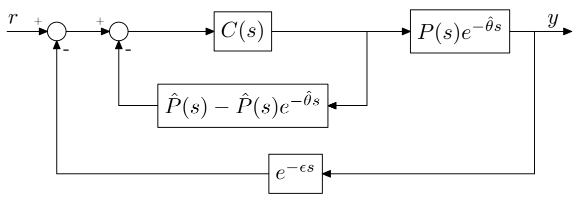


Figure 4. System with mis-matched delays

CLAIM:

If $\hat{\theta}$ is chosen as the minimum value of the possible delay and $\mu$ is chosen to satisfy (19), then the controller (23) is robust against a small difference $\varepsilon$ between the assumed delay $\hat{\theta}$ and the actual delay $\theta = \hat{\theta} + \varepsilon$.

*Proof*:

For $s \in \mathbb{C}_0$,

$$
\begin{aligned}
|H(s)| &= \left| \frac{ks^{\mu}P(s)e^{-\hat{\theta}s}}{1+ks^{\mu}P(s)(1-e^{-\hat{\theta}s})} \right| \\
&\leq \frac{k|1-e^{-2s^{\frac{\alpha}{2}}}||e^{-\theta s}|}{|s^{(\frac{\alpha}{2}-\mu)}(1+e^{-2s^{\frac{\alpha}{2}}})+k(1-e^{-2s^{\frac{\alpha}{2}}})(1-e^{-\theta s})|} \\
&< \frac{k|1-e^{-2s^{\frac{\alpha}{2}}}|}{\left||s^{(\frac{\alpha}{2}-\mu)}(1+e^{-2s^{\frac{\alpha}{2}}})|-k|(1-e^{-2s^{\frac{\alpha}{2}}})(1-e^{-\theta s})|\right|}
\end{aligned}
$$

When $|s| \to \infty$,

$$|s^{(\frac{\alpha}{2}-\mu)}(1+e^{-2s^{\frac{\alpha}{2}}})| \to \infty,$$

while both $|1-e^{-2s^{\frac{\alpha}{2}}}|$ and $|(1-e^{-2s^{\frac{\alpha}{2}}})(1-e^{-\theta s})|$ are bounded.

So

$$\limsup_{|s|\to\infty, s\in\mathbb{C}_0} |H(s)| = 0 < 1.$$

*Remarks*:

In *Theorem 1*, $\varepsilon$ is positive. To satisfy this condition, $\hat{\theta}$ should be chosen as the minimal value of the possible delay.

## 5 Concluding Remarks

In boundary stabilization of the fractional wave equation, well-designed fractional order controllers are robust against a small delay in boundary measurement; while the integer order controller is unstable with an arbitrarily small delay introduced in boundary measurement. For large delays which makes the system unstable, the fractional order controller combined with the Smith predictor is able to compensate the time delay and robust against a small difference between the assumed delay and the actual delay.

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