Proceedings of the IEEE
International Conference on Mechatronics & Automation
Niagara Falls, Canada • July 2005

A Fractional Order PID Tuning Algorithm for A Class of Fractional Order Plants

Chunna Zhao and Dingyü Xue
Faculty of Information Science and Engineering
Northeastern University
Shenyang 110004, P. R. China
xue_dy@21cn.com

YangQuan Chen
CSOIS, Dept. of Electrical & Computer Engineering
Utah State University
Logan, UT 84322-4160, USA
yqchen@ece.usu.edu

***Abstract*— Fractional order dynamic model could model various real materials more adequately than integer order ones and provide a more adequate description of many actual dynamical processes. Fractional order controller is naturally suitable for these fractional order models. In this paper, a fractional order PID controller design method is proposed for a class of fractional order system models. Better performance using fractional order PID controllers can be achieved and is demonstrated through two examples with a comparison to the classical integer order PID controllers for controlling fractional order systems.**

***Index Terms*— Fractional order calculus, fractional order controller, fractional order systems, $PI^{\lambda}D^{\mu}$ controller.**

I. Introduction

The concept of extending classical integer order calculus to non-integer order cases is by no means new. For example, it was mentioned in [1] that the earliest systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville, Riemann, and Holmgren. The most common applications of fractional order differentiation can be found in [2]. The concept has attracted the attention of researchers in applied sciences as well. There has been a surge of interest in the possible engineering application of fractional order differentiation. Examples may be found in [3] and [4]. Some applications including automatic control are surveyed in [5].

In the field of system identification, studies on real systems have revealed inherent fractional order dynamic behavior. The significance of fractional order control is that it is a generalization of classical integral order control theory, which could lead to more adequate modelling and more robust control performance. Reference [6] put forward simple tuning formulas for the design of PID controllers. Some MATLAB tools of the fractional order dynamic system modelling, control and filtering can be found in [13]. Reference [7] gives a fractional order PID controller by minimizing the integral of the error squares. Some numerical examples of the fractional order were presented in [8]. In [9], a PI^{α} controller was designed to ensure that the closed-loop system is robust to gain variations and the step responses exhibit an iso-damping property. For speed control of two-inertia systems, some experimental results were presented in [12] by using a fractional order $PI^{\alpha}D$ controller. A comparative introduction of four fractional order controllers can be found in [10].

In most cases, however, researchers consider the fractional order controller applied to the integer order plant to enhance the system control performance. Fractional order systems could model various real materials more adequately than integer order ones and thus provide an excellent modelling tool in describing many actual dynamical processes. It is intuitively true, as also argued in [11], that these fractional order models require the corresponding fractional order controllers to achieve excellent performance. In this paper, a fractional order PID controller is used to control a class of fractional order systems. A fractional order PID controller design method is proposed with two illustrative examples.

The remaining part of this paper is organized as follows: in Sec. II, mathematical foundation of fractional order controller is briefly introduced; in Sec. III, the fractional order PID controller and its property are presented; in Sec. IV, the fractional order PID controller parameter setting is proposed with specified gain and phase margins; in Sec. V, two examples are presented to illustrate the superior performance achieved by using fractional order controllers. Finally, conclusions are drawn in Sec. VI.

II. A Brief Introduction to Fractional Order Calculus

A commonly used definition of the fractional differointegral is the Riemann-Liouville definition

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} \mathrm{d}\tau \quad (1)$$

for $m-1<\alpha<m$ where $\Gamma(\cdot)$ is the well-known Euler's gamma function. An alternative definition, based on the concept of fractional differentiation, is the Grünwald-Letnikov definition given by

$$ {}_aD_t^\alpha f(t)=\lim_{h\to 0}\frac{1}{\Gamma(\alpha)h^\alpha}\sum_{k=0}^{(t-a)/h}\frac{\Gamma(\alpha+k)}{\Gamma(k+1)}f(t-kh). \quad (2)$$

One can observe that by introducing the notion of fractional order operator ${}_aD_t^\alpha f(t)$, the differentiator and integrator can be unified.

Another useful tool is the Laplace transform. It is shown in [14] that the Laplace transform of an n-th derivative ($n\in R_+$) of a signal $x(t)$ relaxed at $t=0$ is given by: $L\{D^n x(t)\}=s^nX(s)$. So, a fractional order differential equation, provided both the signals $u(t)$ and $y(t)$ are relaxed at $t=0$, can be expressed in a transfer function form

$$G(s)=\frac{a_1s^{\alpha_1}+a_2s^{\alpha_2}+\cdots+a_{m_A}s^{\alpha_{m_A}}}{b_1s^{\beta_1}+b_2s^{\beta_2}+\cdots+b_{m_B}s^{\beta_{m_B}}} \quad (3)$$

where $(a_m,b_m)\in R^2,(\alpha_m,\beta_m)\in R_+^2,\forall(m\in N)$.

III. Basic Ideas of Fractional Order PID Controller

The most common form of a fractional order PID controller is the $PI^\lambda D^\mu$ controller [5], involving an integrator of order λ and a differentiator of order μ where λ and μ can be any real numbers. The transfer function of such a controller has the following form

$$G_c(s)=K_P+\frac{K_I}{s^\lambda}+K_Ds^\mu, \qquad (\lambda,\mu>0). \quad (4)$$

The control signal $u(t)$ can then be expressed in the time domain as

$$u(t)=K_P\ e(t)+K_ID^{-\lambda}\ e(t)+K_DD^\mu\ e(t). \quad (5)$$

Clearly, selecting $\lambda=1$ and $\mu=1$, a classical PID controller can be recovered. Using $\lambda=1$, $\mu=0$, and $\lambda=0$, $\mu=1$, respectively, corresponds to the conventional PI & PD controllers. All these classical types of PID controllers are special cases of the $PI^\lambda D^\mu$ controller given by (4). It can be expected that the $PI^\lambda D^\mu$ controller may enhance the systems control performance. One of the most important advantages of the $PI^\lambda D^\mu$ controller is the possible better control of fractional order dynamical systems. Another advantage lies in the fact that the $PI^\lambda D^\mu$ controllers are less sensitive to changes of parameters of a controlled system [5]. This is due to the two extra degrees of freedom to better adjust the dynamical properties of a fractional order control system. It is pointed out in [15] that a band-limit implementation of fractional order controller is important in practice, and the finite dimensional approximation of the fractional order controller should be done in a proper range of frequencies of practical interest.

IV. A Fractional Order PID Controller Design Method

According to the desired gain margin A_m and phase margin ϕ_m, the designed fractional order PID controller should meet the stability robustness of the feedback control loop. From the basic definitions of gain and phase margin, the dynamic system $G_p(s)$ and the controller $G_c(s)$ should satisfy the following:

$$\phi_m=\arg[G_c(j\omega_g)G_p(j\omega_g)]+\pi \quad (6)$$

$$A_m=\frac{1}{|G_c(j\omega_p)G_p(j\omega_p)|} \quad (7)$$

where ω_g is given by

$$|G_c(j\omega_g)G_p(j\omega_g)|=1 \quad (8)$$

and ω_p satisfies

$$\arg[G_c(j\omega_p)G_p(j\omega_p)]=-\pi. \quad (9)$$

In this paper, we restrict our attention to a class of fractional order plant $G_p(s)$ described by the following transfer function:

$$G_p(s)=\frac{1}{a_1s^\alpha+a_2s^\beta+a_3} \quad (10)$$

where α and β may be any real numbers, and the same to the coefficient $a_1,\ a_2,\ a_3$.

Using $G_c(s)$ as in (4) and $G_p(s)$ as in (10), the following relationships can be established:

$$\begin{aligned}&K_P+\frac{K_I}{\omega_p^\lambda}\cos\frac{\pi\lambda}{2}+K_D\omega_p^\mu\cos\frac{\pi\mu}{2}\\ =&\ -\frac{a_1}{A_m}\omega_p^\alpha\cos\frac{\pi\alpha}{2}-\frac{a_2}{A_m}\omega_p^\beta\cos\frac{\pi\beta}{2}-\frac{a_3}{A_m}; \quad (11)\end{aligned}$$

$$\begin{aligned}&-\frac{K_I}{\omega_p^\lambda}\sin\frac{\pi\lambda}{2}+K_D\omega_p^\mu\sin\frac{\pi\mu}{2}\\ =&\ -\frac{a_1}{A_m}\omega_p^\alpha\sin\frac{\pi\alpha}{2}-\frac{a_2}{A_m}\omega_p^\beta\sin\frac{\pi\beta}{2}; \quad (12)\end{aligned}$$

$$\begin{aligned}&K_P+\frac{K_I}{\omega_g^\lambda}\cos\frac{\pi\lambda}{2}+K_D\omega_g^\mu\cos\frac{\pi\mu}{2}\\ =&\ -a_1\omega_g^\alpha\cos(\frac{\pi\alpha}{2}+\phi_m)-\\ &\ a_2\omega_g^\beta\cos(\frac{\pi\beta}{2}+\phi_m)-a_3\cos\phi_m; \quad (13)\end{aligned}$$

$$-K_I \frac{\sin\frac{\pi\lambda}{2}}{\omega_g^\lambda} + K_D \sin\frac{\pi\mu}{2}\omega_g^\mu$$
$$= -a_1\omega_g^\alpha \sin(\frac{\pi\alpha}{2}+\phi_m) - a_2\omega_g^\beta \sin(\frac{\pi\beta}{2}+\phi_m) - a_3\sin\phi_m. \quad (14)$$

Here, in our controller design problem, the plant model $G_p(s)$ and the expected loop gain and phase margin A_m and ϕ_m are assumed to be known. However, we only have four equations but with seven unknowns $(\omega_p, \omega_g, \lambda, \mu, K_I, K_P, K_D)$. A good news is that, the unknown variables λ, μ, ω_p and ω_g should satisfy the following constraints:

$$(\omega_g^{\lambda+\mu} - \omega_p^{\lambda+\mu})\{a_1[\omega_g^\alpha \cos(\frac{\pi\alpha}{2}+\phi_m) - \omega_p^\alpha \cos\frac{\pi\alpha}{2}]$$
$$+a_2[\omega_g^\beta \cos(\frac{\pi\beta}{2}+\phi_m) - \omega_p^\beta \cos\frac{\pi\beta}{2}]$$
$$+a_3(\cos\phi_m - \frac{1}{A_m})\}$$
$$+(\cot\frac{\pi\lambda}{2} + \cot\frac{\pi\mu}{2})(\frac{\omega_p^\lambda\omega_g^\mu I_p}{A_m} + \omega_g^\lambda\omega_p^\mu I_g)$$
$$-(\frac{\omega_p^{\lambda+\mu} I_p}{A_m} + \omega_g^{\lambda+\mu} I_g)\cot\frac{\pi\mu}{2}$$
$$-(\omega_p^{\lambda+\mu} I_g + \frac{\omega_g^{\lambda+\mu} I_p}{A_m})\cot\frac{\pi\lambda}{2} = 0 \quad (15)$$

where

$$I_p = a_1\omega_p^\alpha \sin\frac{\pi\alpha}{2} + a_2\omega_p^\beta \sin\frac{\pi\beta}{2}, \quad (16)$$

$$I_g = a_1\omega_g^\alpha \sin(\frac{\pi\alpha}{2}+\phi_m) + a_2\omega_g^\beta \sin(\frac{\pi\beta}{2}+\phi_m) + a_3\sin\phi_m. \quad (17)$$

Under these constraints, the parameters λ, μ, ω_p and ω_g can be searched using suitable optimization algorithms. If the parameters $\omega_p, \omega_g, \lambda, \mu$ are given, the controller gains K_I, K_P, K_D can be uniquely decided as follows:

$$K_P = a_1\omega_p^{\lambda+\alpha}\frac{\sin\frac{\pi(\alpha-\mu)}{2}}{A_m\sin(\pi\mu/2)} + a_1\omega_g^{\lambda+\alpha}\frac{\sin[\frac{\pi(\mu-\alpha)}{2}-\phi_m]}{\sin(\pi\mu/2)} + a_2\omega_p^{\lambda+\beta}\frac{\sin\frac{\pi(\beta-\mu)}{2}}{A_m\sin(\pi\mu/2)} + a_2\omega_g^{\lambda+\beta}\frac{\sin[\frac{\pi(\mu-\beta)}{2}-\phi_m]}{\sin(\pi\mu/2)} + a_3\omega_g^\lambda\frac{\sin(\frac{\pi\mu}{2}-\phi_m)}{\sin(\pi\mu/2)} - \frac{a_3\omega_p^\lambda}{A_m}; \quad (18)$$

$$K_I = \frac{\omega_g^\lambda\omega_p^\lambda(A_m\omega_g^\mu I_p - \omega_p^\mu I_g)}{\sin\frac{\pi\lambda}{2}(\omega_g^{\lambda+\mu} - \omega_p^{\lambda+\mu})}; \quad (19)$$

$$K_D = \frac{A_m\omega_p^\lambda I_p - \omega_g^\lambda I_g}{\sin\frac{\pi\mu}{2}(\omega_g^{\lambda+\mu} - \omega_p^{\lambda+\mu})}. \quad (20)$$

Therefore, the fractional order PID controller can be designed according to performance specifications.

V. Two Illustrative Examples

A. Example 1

This example of a heating furnace was considered in [16], which can be modelled by the integer and the fractional order differential equations, respectively. According to [16], the integer order model (IOM) of the heating furnace is a second order transfer function

$$G_{Ip}(s) = \frac{1}{73043s^2 + 4893s + 1.93} \quad (21)$$

while the fractional order model (FOM) is given by:

$$G_{Fp}(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69}. \quad (22)$$

The unit-step responses of the heating furnace models (IOM and FOM) are compared in Fig. 1 where it was remarked in [16] that the fractional order model is more exact than the integer order model.

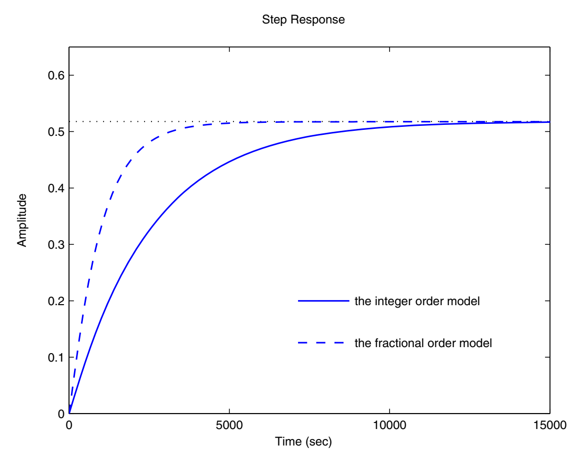

Fig. 1. Comparison of unit step responses of the integer order model and the fractional order model of the heating furnace

To design an integer order PID controller, the integer order model (21) should be approximated by a first order lag plus time delay system which is given in the following:

$$G_{Ip}(s) = \frac{0.51813}{2520.2609s + 1}e^{-14.97s}.$$

Then, according to the Åström-Hägglund tuning algorithm [17], an integer order PID controller is designed

$$G_{Ic}(s) = 305.38 + \frac{10.18}{s} + 2290.35s. \qquad (23)$$

Figure 2 shows the results of the same conventional PID controller (23) applied to the integer order model (21) and the fractional order model (22). We can observe that using an integer PID controller for the fractional order system, the performance becomes worse than for the integer order model. Note that both cases exhibit long settling time and large overshoot.

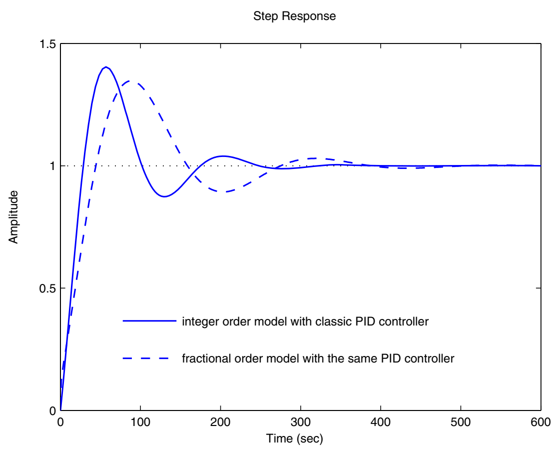

Fig. 2. Comparison of unit step responses of the closed loop integer order model and the closed loop fractional order system with the same integer order controller (23)

Now, let us consider the $PI^{\lambda}D^{\mu}$ controller. Here, the expected loop phase and gain margins are specified as $\phi_m = 60°$ and $A_m = 1.1$. Selecting $\lambda = 0.6$ and $\mu = 0.35$, the fractional order PID controller can be designed by the proposed method in this paper. Thus, using our tuning algorithm in the paper, one can have $K_p = 736.8054$, $K_i = -0.5885$, and $K_d = -818.4204$. The fractional order PID controller is then designed as

$$G_{Fc}(s) = 736.8054 - \frac{0.5885}{s^{0.6}} - 818.4204s^{0.35} \qquad (24)$$

The step response of the fractional order PID controller applied to the fractional order model is given in Fig. 3. The corresponding Bode diagrams of the controlled models are presented in Fig. 4. It can be seen that there is an obvious improvement, i.e., with a much faster system response and also a smaller overshoot when the fractional order PID controller is applied to the fractional order model.

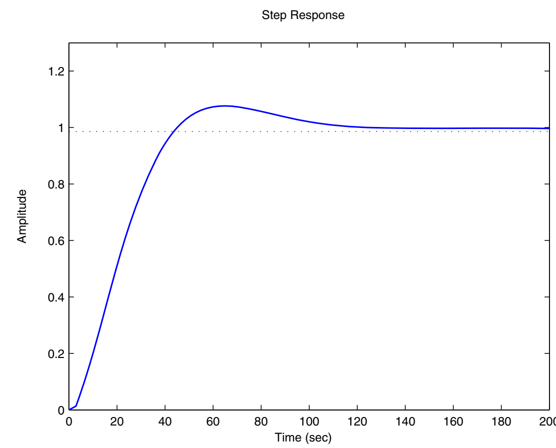

Fig. 3. Step response of the closed loop fractional order model with fractional order controller

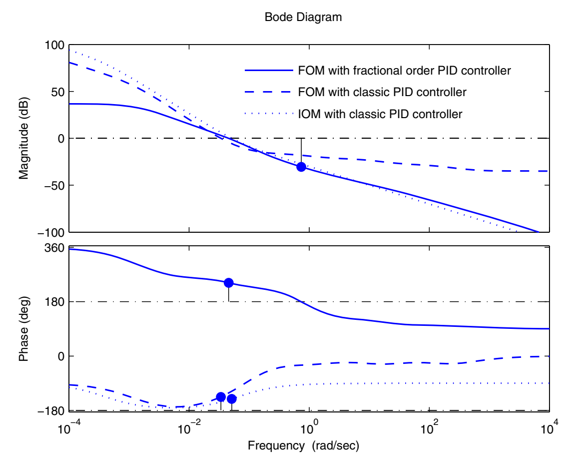

Fig. 4. Comparison of Bode diagrams of the fractional order model with the fractional order $PI^{\lambda}D^{\mu}$ controller, and the same model with the integer order controller, and the integer order model with the integer order controller

B. Example 2

We consider the following fractional order plant model given in [14]:

$$G_{Fp}(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1}. \qquad (25)$$

Using the least-squares method, the following approximating integer order model corresponding to (25) was obtained in [14]:

$$G_{Ip}(s) = \frac{1}{0.7414s^2 + 0.2313s + 1}. \qquad (26)$$

The comparison of the unit step responses of the systems described by (25) and (26) are shown in Fig. 5.

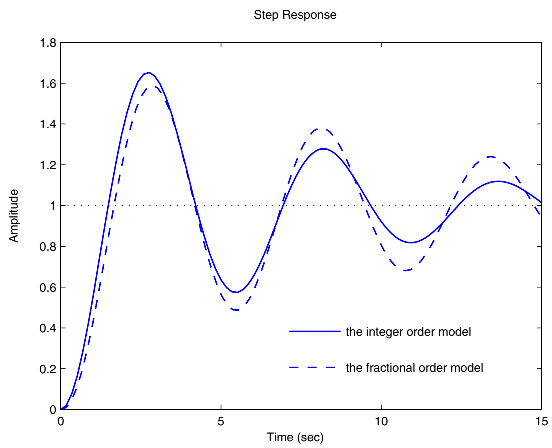

Fig. 5. Comparison of unit step response of the integer order model and the fractional order system

The integer order PD controller and the fractional order PD$^{\mu}$ controller were designed in [14]. The integer order PD controller was given by

$$G_c(s) = 20.5 + 2.7343s \tag{27}$$

while the fractional order PD$^{\mu}$ controller characterized by the fractional order transfer function [14]

$$G_c(s) = 20.5 + 3.7343s^{1.15}. \tag{28}$$

In Fig. 6, comparison of the unit step responses of the closed loop fractional order system controlled by fractional order PD$^{\mu}$ controller and the integer order PD controller is given. The conclusion was that the use of the fractional order controller leads to an improvement of the control of the fractional order system. It should be pointed out, however, since there is no integral action introduced, zero steady state error cannot be achieved.

According to the proposed method, by trial and error, one can select $\lambda = 0.1$ and $\mu = 1.15$. When one selects $\phi_m = 60°$ and $A_m = 1.3$, the controller parameters $K_p = 233.4234$, $K_i = 22.3972$, and $K_d = 18.5274$ can then be obtained. The fractional order PID controller is

$$G_c(s) = 233.4234 + \frac{22.3972}{s^{0.1}} + 18.5274s^{1.15}. \tag{29}$$

The unit step response of the closed loop with the fractional order system (25) controlled by fractional order PI$^{\lambda}$D$^{\mu}$ controller (29) is given in Fig. 7. The settling time of the closed loop system is 0.573 second which is much shorter than those of the existing PD controllers. The corresponding Bode diagrams of the fractional order model with the fractional order PD$^{\mu}$ controller, and with the fractional order PI$^{\lambda}$D$^{\mu}$ controller are presented in Fig. 8.

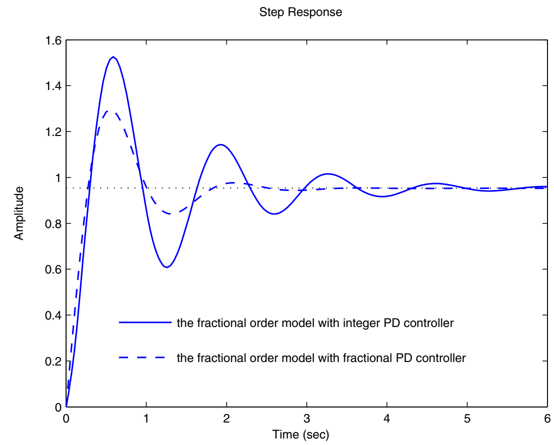

Fig. 6. Comparison of unit step responses of the closed loop fractional order model with the integer order PD controller and with fractional order PD$^{\mu}$ controller

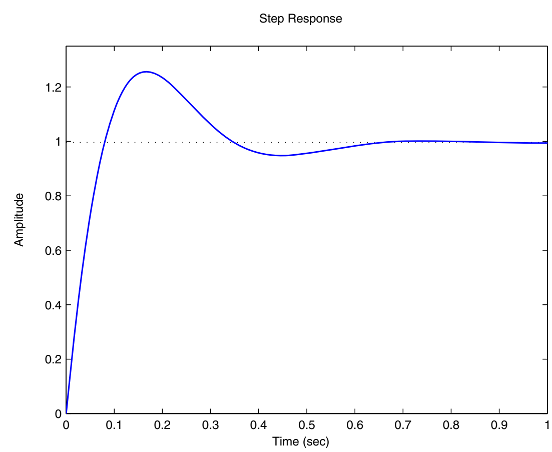

Fig. 7. Step response of the closed loop fractional order model with fractional order PID controller

It can be seen that the fractional order PI$^{\lambda}$D$^{\mu}$ controller designed by the proposed method in this paper are more effective for the fractional order systems.

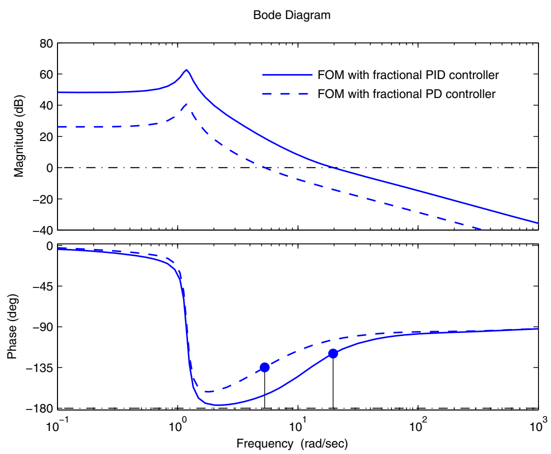

Fig. 8. Comparison of bode diagrams of the fractional order model with the fractional order PD^{μ} controller, and with the fractional order $PI^{\lambda}D^{\mu}$ controller

VI. Conclusion

In this paper, a fractional order PID controller design method is proposed for the fractional order system model. The benefits of fractional order models for real dynamical objects and processes become more and more obvious. Through two fractional order dynamical models, the fractional order PID controller design by the proposed method has been demonstrated. The simulation results illustrate that fractional order PID controller achieves better control performance with the proposed design method.

References

[1] K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York and London, 1974.

[2] M. Axtell and M.E. Bise, "Fractional calculus applications in control systems," the IEEE National Aerospace and Electronics Conference, New York, USA, pp. 563-566, 1990.

[3] N. Engheta, "On fractional calculus and fractional multipoles in lectromagnetism," IEEE Trans. Antennas Propagat, vol. 44, pp. 554-566, 1996.

[4] M. Unser and T. Blu, "Wavelet theory demystified," IEEE Trans. Signal Processing, vol. 51, pp. 470-483, 2003.

[5] I. Podlubny, "Fractional-order systems and $PI^{\lambda}D^{\mu}$ -controllers," IEEE Trans. Automatic Control, vol. 44, pp. 208-214, 1999.

[6] W.K. Ho, K.W. Lim, and W. Xu, "Optimal gain and phase margin tuning for PID controllers," Automatica, Vol.34, No.8, pp.1009-1014, 1998.

[7] Z.F. Lv, "Time-domain simulation and design of SISO feedback control systems", Doctoral Dissertation, National Cheng Kung University, 2004, Taiwan, China.

[8] R. Caponetto, L. Fortuna, and D. Porto, "A new tuning strategy for a non integer order PID controller," IFAC2004, Bordeaux, France, 2004.

[9] C.A. Monje, B.M. Vinagre, Y.Q. Chen, V. Feliu, P. Lanusse and J. Sabatier, "Proposals for Fractional $PI^{\lambda}D^{\mu}$ Tuning", The First IFAC Symposium on Fractional Differentiation and its Applications 2004, Bordeaux, France, July 19-20, 2004.

[10] Dingyü and YangQuan Chen, "A Comparative Introduction of Four Fractional Order Controllers". Proc. of The 4th IEEE World Congress on Intelligent Control and Automation (WCICA02), June 10-14, 2002, Shanghai, China. pp. 3228-3235.

[11] YangQuan Chen, Dingyu Xue, and Huifang Dou. "Fractional Calculus and Biomimetic Control". IEEE Int. Conf. on Robotics and Biomimetics (RoBio04), August 22-25, Shengyang, China. (PDF-robio2004-347)

[12] C. B. Ma and Y. Hori, "Design of fractional order PID controller for robust two-inertia speed control to torque saturation and load inertia variation," IPEMC 2003, Xi'an, China.

[13] Dingyü and YangQuan Chen, "*Advanced Mathematic Problem Solution Using MATLAB*" (in Chinese, ISBN 7-302-09311-3/O.392), Beijing: Tsinghua University Press, 2004.

[14] I. Podlubny, "Fractional-order Systems and Fractional-order Controllers", The Academy of Sciences Institute of Experimental Physics, UEF-03-94, Kosice, Slovak Republic, 1994.

[15] A. Oustaloup, F. Levron, B. Mathieu, and F.M. Nanot, "Frequency-band complex noninteger differentiator: characterization and synthesis," IEEE Trans. Circuits Syst. I, vol. 47, pp. 25-39, 2000.

[16] I. Podlubny, L. Dorcak, and I. Kostial, "On fractional derivatives, fractional-order dynamic system and $PI^{\lambda}D^{\mu}$ -controllers", Proc. of the 36th IEEE CDC, San Diego,1999.

[17] K. J. Åström and T. Hägglund, PID Controllers: Theory, Design and Tuning, Research Triangle Park, Instrument Society of America, 1995.