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Monotonic Convergent Iterative Learning Controller Design based on Interval Model Conversion

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***Abstract*— This paper presents the design of a robust iterative learning controller for the case of a plant with interval model uncertainty in the "A-matrix" of its state space description. First order perturbation theory is utilized to find bounds on the eigenvalues and eigenvectors of the powers of A when A is an interval matrix. These bounds are then used for calculation of the interval uncertainty of the Markov matrix, which can then be used to design an iterative learning controller that ensures monotonic convergence for all systems in the interval plant.**

***Index Terms*— Iterative learning control, interval conversion, matrix perturbation, monotonic convergence.**

I. INTRODUCTION

Iterative learning control (ILC) is a control strategy for systems that execute the same trajectory, motion, or operation over and over. ILC exploits the repetitiveness inherent in such processes to improve their performance from trial to trial [1], [2]. Generally, ILC can lead to convergence even when the system is poorly modelled, but still some critical requirements must be met. For example, in a linear plant asymptotic stability is achieved if and only if $|1 - \gamma h_1| < 1$, where h_1 is the first Markov parameter and γ is the learning gain. Most existing ILC works have focused on optimality and analytical solution for a nominal h_1 under the assumption that h_1 has no uncertainty. In this paper our concern is to relax this assumption and to design the learning gain matrix while considering model uncertainties in the plant. In particularly, this paper designs the learning gain matrix on the super-vector framework to guarantee the monotone convergence of the output response on the iteration axis for linear plants in which the A-matrix is an interval matrix.

The paper is organized as follows. In Section II we discuss how to convert from an interval plant modelled in the state space to one modelled using interval Markov parameters. We refer to this as interval model conversion. Next, in Section III matrix perturbation methods are utilized to find bounds on perturbed eigenpairs associated with powers of A and in Section IV bounds on the Markov parameters are computed. Optimization schemes are then suggested to design the iterative learning gain matrix in Section V. Simulation tests and conclusions are given in Section VI and in Section VII, respectively.

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II. INTERVAL MODEL CONVERSION IN ILC

In this section, eigenpair-decomposition is suggested to convert interval uncertainty in the state space model to interval uncertainty in the super-vector ILC model form. In the super-vector framework, described below in Section V and in [3], [4], [5], [6], Markov parameters are used for ILC design. Thus, to accomplish the robust ILC design that we seek to demonstrate in this paper, the interval uncertainty of the nominal plant, either state space or transfer function, has to be converted into interval uncertainties in the Markov parameters. This process is termed "interval model conversion."

In this paper, we focus on the nominal discrete system model given in the following single input single output state space form:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where A, B, and C are the matrices describing the system in the state space; $x(t), u(t)$, and $y(t)$ are the state, input, and output variables, respectively. Particularly, we call A the *nominal plant matrix*. Without loss of generality, the relative degree of the system is assumed to be 1. Then, Markov parameters are calculated by $h_k = CA^{k-1}B$. These Markov parameters form the Markov matrix, which is denoted as H (see Section V below and refer to [1], [7], [8] for notations of ILC and the super-vector ILC framework). Then, the conversion process is to find h_k from uncertain A, B, and C using $h_k = CA^{k-1}B$. To simplify our presentation, it is assumed that the model uncertainty exists in A only. From now on, the superscript I is used such as A^I to include the model uncertainty in A. Then, the conversion process can be defined as the process to find the uncertain boundaries of $h_k^I \in [\underline{h_k}, \overline{h_k}]$ from A^I. Throughout this paper, the underline $\underline{(\cdot)}$ represents the minimum value of $(\cdot)$ and the overline $\overline{(\cdot)}$ represents the maximum value of $(\cdot)$. To clarify these concepts consider the following definitions.

Definition 2.1: The $n \times n$ interval matrix set A^I is defined as

$$A^I = \left\{ A = [a_{ij} : a_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}]] \right\}$$

Definition 2.2: The nominal matrix A_0 is defined to be

$$A_0 = \left[a_{ij}^0 : {a^0}_{ij} = \frac{\underline{a_{ij}} + \overline{a_{ij}}}{2} \right]$$

where $\underline{a_{ij}}$ and $\overline{a_{ij}}$ are the lower and upper bounds associated with the elements a_{ij} of A^I.

Definition 2.3: The interval perturbation matrix set ΔA^I is defined as

$$\Delta A^I = \left\{\Delta A = \left[\Delta a_{ij} : \Delta a_{ij} = \overline{a_{ij}} - {a^0}_{ij} = {a^0}_{ij} - \underline{a_{ij}}\right]\right\}$$

where $\underline{a_{ij}}$, $\overline{a_{ij}}$, and a^0_{ij} are the lower bounds, upper bounds, and nominal values associated with the elements a_{ij} of A^I, respectively.

Note that we can also write

$$\Delta A^I = \left\{\Delta A : \Delta A = A - A_0, \forall A \in A^I\right\}$$

Definition 2.4: The vertex matrix set A^v is a subset of the interval matrix set A^I that is defined as:

$$A^v = \left\{A \in A^I : a_{ij} \in \{\underline{a_{ij}}, \overline{a_{ij}}\}\right\}$$

Next, to proceed we make two assumptions. The first is a technical assumption. Although we conjecture that this assumption can be relaxed, it is easiest to proceed with the following requirement. The second is more practical, as will be described in a later remark.

Assumption 2.1: Any matrix $A \in A^I$ is diagonable and can be decomposed as: $A = X\Lambda X^{-1}$, with $\Lambda = \mathrm{diag}(\lambda_i)$, where λ_i are the eigenvalues of A.

Assumption 2.2: Any matrix $A \in A^I$ is stable.

Throughout this paper, we call X the left eigenvector matrix, Λ the eigenvalue matrix, and $Y := X^{-1}$ the right eigenvector matrix. We also define the interval matrix sets

$$\Lambda_A = \left\{\Lambda : \ A = X\Lambda X^{-1}, A \in A^I\right\}$$

$$X_A = \left\{X : \ A = X\Lambda X^{-1}, A \in A^I\right\}$$

$$Y_A = \left\{Y = X^{-1} : \ X \in X_A\right\}$$

We now define two interval power sets:

Definition 2.5: The set of the power of the interval matrix set is defined as

$$(A^I)^k = \left\{A^k : A \in A^I\right\},$$

where k is the order of power.

Definition 2.6: The set of the generalized power of the interval matrix set is defined as

$$A^I_k = \left\{A = X\Lambda^k Y : \Lambda \in \Lambda_A, X \in X_A, Y \in Y_A\right\},$$

where k is the order of power.

Note that clearly

$$(A^I)^k \subset A^I_k.$$

Remark 2.1: From a computational perspective, it should be noticed that the boundary of the set $(A^I)^k$ can be estimated by multiplying the interval matrices using interval calculation software such as *Intlab* [9], but the result will be quite conservative as k increases and it requires huge amount of computation. Further, notice that the exact boundary of $(A^I)^k$ cannot be calculated mathematically or analytically.

Remark 2.2: On the other hand, since $(A^I)^k \subset A^I_k$, we can estimate the boundary of the set $(A^I)^k$ by estimating the boundary of the set A^I_k, which is also easily done in *Intlab*. Specifically, since the boundary of A^I_k can be estimated by using three different sets: Λ_A, X_A, and Y_A, the lower and upper boundary of $(A^I)^k$ can be subsequently estimated.

Remark 2.3: From the fact that the boundary of A^I_k is estimated from Λ_A, X_A, and Y_A, we observe that if the maximum eigenvalue of $A \in A^I$ is less than 1, for all $\mathcal{A}_k \in A^I_k$, $\mathcal{A}_k$ will converge to zero as $k \to \infty$ because Λ^k, $\Lambda \in \Lambda_A$, converges to zero as $k \to \infty$. However, if the maximum eigenvalue is bigger than 1, any $\mathcal{A}_k \in A^I_k$ will diverge, hence the Markov parameters become bigger and our bound on the the uncertain interval boundary of h_k become bigger as k increases. For this reason we assumed A is stable.

In this section we showed that we can contain our original interval system inside a "bigger" interval system according to: $(A^I)^k \subset A^I_k$. Therefore, the remaining work is to estimate the boundaries of Λ_A, X_A, and Y_A and from these estimates to compute bounds on A^k. The next section suggests an analytical method to estimate the bounds of Λ_A, X_A, and Y_A from A^I using first-order perturbation theory [10].

III. INTERVAL MATRIX EIGENPAIR BOUNDS

In this section, we briefly summarize first order perturbation theory and then suggest two lemmas to obtain analytical solutions of the boundaries of X_A, Λ_A and Y_A from A^I to be used for estimating the boundary of A^I_k

Let us suppose that the $n \times n$ nominal matrix A_0 has its n different nominal eigenvalues λ_{0i}, nominal left eigenvectors x_{0i}, and nominal right eigenvectors y_{0i}, where $i = 1, \cdots, n$, defined by $x^T_{0i}A_0 = \lambda_{i0}x^T_{0i}$ and $A_0 y_{0i} = \lambda_{i0}y_{0i}$. Further define the nominal eigenvalue matrix, Λ_0, nominal left eigenvector matrix, X_0, and nominal right eigenvector matrix, Y_0 by $\Lambda_0 = [\lambda_{ij} : \lambda_{ij} = \lambda_{0i} \text{ if } i = j, \ \lambda_{ij} = 0 \text{ if } i \neq j]$, $X_0 = [x_{01}, x_{02}, \cdots, x_{0n}]^T$, $Y_0 = [y_{01}, y_{02}, \cdots, y_{0n}]$, and with $Y_0^{-1} = X_0$. Next, assume a small perturbation in A_0 such as $A = A_0 + \Delta A$, where $A \in A^I$ and $\Delta A \in \Delta A^I$ based on Definition 2.1 and Definition 2.3. Denote the eigenvalues and eigenvectors associated with the perturbation ΔA as: λ_{1i}, x_{1i}, and y_{1i}, respectively. In other words, when λ_i, x_i, and y_i represent eigenvalues and eigenvectors of $A \in A^I$, the following relationships are satisfied: $\lambda_i = \lambda_{0i} + \lambda_{1i}$, $x_i = x_{0i} + x_{1i}$, and $y_i = y_{0i} + y_{1i}$[1]. Observe that λ_{1i} are scalar intervals, and x_{1i} and y_{1i} are interval vectors. In this section, we are interested in finding the boundaries of λ_{1i}, x_{1i}, and y_{1i}, because we can only estimate boundaries of λ_i, x_i, and y_i by estimating boundaries of λ_{1i}, x_{1i}, and y_{1i}.

From [10], the following formulae are adopted for the perturbed eigenvalues:

$$\lambda_{1i} = x^T_{0i}\Delta A y_{0i}, \ \forall \ \Delta A \in \Delta A^I \tag{2}$$

and for the perturbed eigenvectors:

$$x_{1i} = \sum_{k=1}^{n} \gamma_{ik}x_{0k}; \ y_{1i} = \sum_{k=1}^{n} \varepsilon_{ik}y_{0k}, \tag{3}$$

where

$$\gamma_{ik} = \frac{y^T_{0k}\Delta A x_{0i}}{\lambda_{0i} - \lambda_{0k}}; \ \varepsilon_{ik} = \frac{x^T_{0k}\Delta A y_{0i}}{\lambda_{0i} - \lambda_{0k}},$$

$$i, k = 1, \cdots, n; i \neq k, \ \forall \ \Delta A \in \Delta A^I,$$

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[1] Note that we do not imply that the eigenvalues of the sum of two matrices are equal to the sum of the individual matrices. Rather, the eigenpairs λ_{1i}, x_{1i}, and y_{1i} are "perturbation" eigenpairs and represent what would be added to the nominal values to obtain equivalent eignepairs of the perturbed matrix. Also note, on page 103 of [10], λ_{1i}, x_{1i}, and y_{1i} are called the first order perturbation eigensolution.

but when $i = k$, $\gamma_{ik} = 0$ and $\varepsilon_{ik} = 0$. However, here, notice that $\Delta A \in \Delta A^I$, where ΔA^I is an interval perturbation matrix set, so it is quite messy to calculate λ_{1i}, x_{1i}, and y_{1i} in (2) and (3). For reliable analytical calculation of (2) and (3), the maximum absolute values of the real and imaginary parts are considered separately. Let us denote the maximum of λ_{1i} in real axis by $\lambda_{1i}|_{real}^{\max}$, which can be defined as $\lambda_{1i}|_{real}^{\max} = \max\{\lambda_{1i}|_{real} : \lambda_{1i}|_{real} = |\mathrm{Re}(\lambda_{1i})|,\ \lambda_{1i} = x_{0i}^T \Delta A y_{0i},\ \forall\ \Delta A \in \Delta A^I\}$, and the maximum of λ_{1i} in imaginary axis by $\lambda_{1i}|_{imag}^{\max}$, which can be defined as $\lambda_{1i}|_{imag}^{\max} = \max\{\lambda_{1i}|_{imag} : \lambda_{1i}|_{imag} = |\mathrm{Im}(\lambda_{1i})|,\ \lambda_{1i} = x_{0i}^T \Delta A y_{0i},\ \forall\ \Delta A \in \Delta A^I\}$, where Re means the real part and Im means the imaginary part. Then, the following lemma is suggested.

Lemma 3.1: Considering the real part and imaginary part separately, at any fixed i in above definition (i.e., at a fixed eigenvalue), the followings are true:

$$\lambda_{1i}|_{real}^{\max} = \max_{(\Delta A)^v \subset \Delta A^I} \{\mathrm{Re}[x_{0i}^T (\Delta A)^v y_{0i}]\}, \tag{4}$$

$$\lambda_{1i}|_{imag}^{\max} = \max_{(\Delta A)^v \subset \Delta A^I} \{\mathrm{Im}[x_{0i}^T (\Delta A)^v y_{0i}]\}, \tag{5}$$

where $(\Delta A)^v$ are the vertex matrices of ΔA^I.

Proof: The proof is included in the proof of Lemma 3.2. ■

The perturbed radii of eigenvectors also can be estimated using the vertex matrix finite set. First, let us consider the left eigenvectors and let us denote j^{th} element of x_{1i} by $(x_{1i})_j$. Then, denoting the maximum of $(x_{1i})_j$ in real axis by $(x_{1i})_j|_{real}^{\max}$, which can be defined as $(x_{1i})_j|_{real}^{\max} = \max\{(x_{1i})_j|_{real} : (x_{1i})_j|_{real} = |\mathrm{Re}((x_{1i})_j)|,\ (x_{1i})_j = \sum_{k=1}^n \frac{y_{0k}^T \Delta A x_{0i}}{\lambda_{0i} - \lambda_{0k}} (x_{0k})_j,\ \forall\ \Delta A \in \Delta A^I\}$, and denoting the maximum of $(x_{1i})_j$ in imaginary axis by $(x_{1i})_j|_{imag}^{\max}$, which can be defined as $(x_{1i})_j|_{imag}^{\max} = \max\{(x_{1i})_j|_{imag} : (x_{1i})_j|_{imag} = |\mathrm{Re}((x_{1i})_j)|,\ (x_{1i})_j = \sum_{k=1}^n \frac{y_{0k}^T \Delta A x_{0i}}{\lambda_{0i} - \lambda_{0k}} (x_{0k})_j,\ \forall\ \Delta A \in \Delta A^I\}$, we provide the following lemma (the perturbed radii of right-eigenvectors are calculated in the same way).

Lemma 3.2: Considering the real part and imaginary part separately, at any fixed i and fixed j (i.e., at a fixed element of the fixed eigenvector), the maximum perturbation of $(x_{1i})_j$ can be calculated by checking the vertex matrices of interval perturbation matrix set such as:

$$(x_{1i})_j|_{real}^{\max} = \max_{(\Delta A)^v \subset \Delta A^I} \left\{ \mathrm{Re}\left[\sum_{k=1}^n \frac{y_{0k}^T (\Delta A)^v x_{0i}}{\lambda_{0i} - \lambda_{0k}} (x_{0k})_j \right] \right\}$$

$$(x_{1i})_j|_{imag}^{\max} = \max_{(\Delta A)^v \subset \Delta A^I} \left\{ \mathrm{Im}\left[\sum_{k=1}^n \frac{y_{0k}^T (\Delta A)^v x_{0i}}{\lambda_{0i} - \lambda_{0k}} (x_{0k})_j \right] \right\},$$

where $(x_{0k})_j$ is the j^{th} element of the k^{th} eigenvector x_{0k}.

Proof: From (3), we have

$$\begin{aligned} x_{1i} &= \gamma_{i1} x_{01} + \cdots + \gamma_{in} x_{0n} \\ &= \frac{y_{01}^T \Delta A x_{0i}}{\lambda_{0i} - \lambda_{01}} x_{01} + \cdots + \frac{y_{0n}^T \Delta A x_{0i}}{\lambda_{0i} - \lambda_{0n}} x_{0n} \end{aligned} \tag{6}$$

In (6), since the denominators $\lambda_{0i} - \lambda_{01}, \cdots, \lambda_{0i} - \lambda_{0n}$ are scalars and $y_{01}, \cdots, y_{0n}, x_{0i}, x_{01}, \cdots, x_{0n}$ are vectors, (6) is rewritten such as:

$$x_{1i} = \xi^1 \Delta A \eta^1 x_{01} + \cdots + \xi^n \Delta A \eta^n x_{0n}, \tag{7}$$

where the substitutions $\frac{x_{0i}}{\lambda_{0i} - \lambda_{01}} = \eta^1, \cdots, \frac{x_{0i}}{\lambda_{0i} - \lambda_{0n}} = \eta^n$, and $y_{01}^T = \xi^1, \cdots, y_{0n}^T = \xi^n$ are used. Observing that $\xi^1 \Delta A \eta^1, \cdots,, \xi^n \Delta A \eta^n$ are scalars, we consider the j^{th} element of the vector x_{1i} such as:

$$(x_{1i})_j = \xi^1 \Delta A \eta^1 (x_{01})_j + \cdots + \xi^n \Delta A \eta^n (x_{0n}) j. \tag{8}$$

Let us rewrite (8) as:

$$\begin{aligned} (x_{1i})_j &= \left\{ \sum_{k=1}^n \sum_{l=1}^n \left((\xi^1)_k (\Delta A)_{kl} (\eta^1)_l \right) \right\} (x_{01})_j \\ &\quad \vdots \\ &+ \left\{ \sum_{k=1}^n \sum_{l=1}^n \left((\xi^n)_k (\Delta A)_{kl} (\eta^n)_l \right) \right\} (x_{0n})_j \\ &= \sum_{p=1}^n \left\{ \sum_{k=1}^n \sum_{l=1}^n \left((\xi^p)_k (\Delta A)_{kl} (\eta^p)_l \right) \right\} (x_{0p})_j \\ &= \sum_{k=1}^n \sum_{l=1}^n \left\{ \sum_{p=1}^n \left((\xi^p)_k (\eta^p)_l \right) (x_{0p})_j \right\} (\Delta A)_{kl}. \end{aligned} \tag{9}$$

Therefore, since $\sum_{p=1}^n ((\xi^p)_k (\eta^p)_l)(x_{0p})_j$ is a complex number, simply by denoting

$$\alpha_{kl} + \beta_{kl} \mathrm{i} := \sum_{p=1}^n \left((\xi^p)_k (\eta^p)_l \right) (x_{0p})_j,$$

we have

$$(x_{1i})_j = \sum_{k=1}^n \sum_{l=1}^n (\alpha_{kl} + \beta_{kl} \mathrm{i}) (\Delta A)_{kl}. \tag{10}$$

Now, in order to find the maximum absolute magnitude of $(x_{1i})_j$, we separate the real part and imaginary part. Let us first investigate the maximum absolute value of the real part. The real part is $\sum_{k=1}^n \sum_{l=1}^n \alpha_{kl} (\Delta A)_{kl}$, where $\alpha_{kl} \in \Re$ and $(\Delta A)_{kl}$ are scalar intervals. Observe that $\sum_{k=1}^n \sum_{l=1}^n \alpha_{kl} (\Delta A)_{kl}$ is a scalar interval, which ranges such as $[\underline{\delta}, \overline{\delta}]$, where

$$\underline{\delta} = \min_{(\Delta A)_{kl} = [a_{ij}^0 - \underline{a_{ij}}, \overline{a_{ij}} - a_{ij}^0]} \left\{ \sum_{k=1}^n \sum_{l=1}^n \alpha_{kl} (\Delta A)_{kl} \right\}$$

and

$$\overline{\delta} = \max_{(\Delta A)_{kl} = [a_{ij}^0 - \underline{a_{ij}}, \overline{a_{ij}} - a_{ij}^0]} \left\{ \sum_{k=1}^n \sum_{l=1}^n \alpha_{kl} (\Delta A)_{kl} \right\}.$$

Finally, we find that $\underline{\delta}$ occurs at a vertex point of $(\Delta A)_{kl}$ depending on signs of α_{kl}. In other words, if $\alpha_{kl} \geq 0$, then the minimum of δ occurs at $a_{ij}^0 - \underline{a_{ij}}$; else if $\alpha_{kl} < 0$, then the minimum of δ occurs at $\overline{a_{ij}} - a_{ij}^0$. In the same way, $\overline{\delta}$ occurs at a vertex point of $(\Delta A)_{kl}$ depending on signs of α_{kl}. If $\alpha_{kl} \geq 0$, then the maximum of δ occurs at $\overline{a_{ij}} - a_{ij}^0$; else if $\alpha_{kl} < 0$, then the maximum of δ occurs at $a_{ij}^0 - \underline{a_{ij}}$. Next, the same procedure can be repeated in the imaginary part as the procedure performed in the real part. Thus, the maximum and minimum boundaries of eigenvectors can be checked by investigating the vertex matrices of interval perturbation matrix set. ■

Lemma 3.1 and Lemma 3.2 show how the maximum magnitude of the perturbed eigenvalues and eigenvectors can be calculated, respectively. Thus, since the interval eigenpairs are calculated by $\lambda_i = \lambda_{0i} + \lambda_{1i}$, $x_i = x_{0i} + x_{1i}$ and $y_i = y_{0i} + y_{1i}$, we have effectively computed the bounds on the sets Λ_A, X_A, and Y_A.

IV. MARKOV PARAMETER BOUNDS

In Section II, the interval model conversion method was developed and in Section III, an analytical method for finding the maximum magnitudes of the interval eigenpairs was suggested. That is, Section II showed that the interval boundaries of A^k, where $A \in A^I$, can be bounded using the inequality: $(A^I)^k \subset A^I_k$, which provides the following relationship:

$$\underline{\mathcal{A}^k} \le \underline{A^k} \le A^k \le \overline{A^k} \le \overline{\mathcal{A}^k} \tag{11}$$

where $A^k \in (A^I)^k$ and $\mathcal{A}^k \in A^I_k$. Then, the interval boundaries of Markov parameters (ex: $h_{k+1} = CA^kB$) can be estimated such as:

$$\underline{h_{k+1}} = \underline{CA^kB}; \ \overline{h_{k+1}} = \overline{CA^kB} \tag{12}$$

where C, B are constant vectors describing the system (1) and A^k are the interval matrices which are lower-bounded and upper-bounded by $\underline{\mathcal{A}^k} \le A^k \le \overline{\mathcal{A}^k}$ from (11). Finally, Lemma 3.1 and Lemma 3.2 of Section III showed that the analytical solution for estimating the boundaries of the interval matrix set $A^I_k = \{A = X\Lambda^kY : \Lambda \in \Lambda_A, X \in X_A, Y \in Y_A\}$ can be obtained using the vertex matrices of interval perturbation matrix set ΔA^I. The following computational procedure summarizes Section II and Section III:

- Define the interval boundaries $\underline{a_{ij}}$ and $\overline{a_{ij}}$.
- Based on Definition 2.3, estimate the associated maximum perturbations of eigenvalues in real part and imaginary part using (4) and (5) of Lemma 3.1. From this procedure, we estimate $\Lambda_A := \{\Lambda : \Lambda = [\lambda_i : \lambda_i = \lambda_{0i} + \lambda_{1i},\ \mathrm{Re}(\lambda_{1i}) = \left[\underline{\mathrm{Re}(\lambda_i)}, \overline{\mathrm{Re}(\lambda_i)}\right],\ \mathrm{Im}(\lambda_{1i}) = \left[\underline{\mathrm{Im}(\lambda_i)}, \overline{\mathrm{Im}(\lambda_i)}\right], i = 1, \cdots, n]\}$, because the extreme boundaries of λ_i can be calculated as
$$\overline{\mathrm{Re}(\lambda_i)} = \mathrm{Re}(\lambda_{0i}) + \lambda_{1i}|^{\max}_{real},$$
$$\underline{\mathrm{Re}(\lambda_i)} = \mathrm{Re}(\lambda_{0i}) - \lambda_{1i}|^{\max}_{real},$$
$$\overline{\mathrm{Im}(\lambda_i)} = \mathrm{Im}(\lambda_{0i}) + \lambda_{1i}|^{\max}_{imag},$$
$$\underline{\mathrm{Im}(\lambda_i)} = \mathrm{Im}(\lambda_{0i}) - \lambda_{1i}|^{\max}_{imag}.$$
- Use Lemma 3.2 to estimate the maximum perturbations of all elements of eigenvectors such as $(x_{1i})_j|^{\max}_{real}$ and $(x_{1i})_j|^{\max}_{imag}$. Repeat this procedure from $i = 1$ to $i = n$ doing the repetition $j = 1$ to $j = n$ in every i. Then, X_A and Y_A are estimated from the same procedures as done in eigenvalue estimation.
- Recall that Λ^k can be calculated easily because $\Lambda \in \Lambda_A$, where Λ_A is a set composed of diagonal matrices. In calculating $A^I_k = \{X\Lambda^kY\}$, the interval algebra may be used, because X Λ^k, and Y are interval matrices (i.e., $X \in X_A$, $\Lambda \in \Lambda_A$, and $Y \in Y_A$). For the multiplication of these three interval matrices, *Intlab* [9] can be used.
- Finally, use $(A^I)^k \subset A^I_k$ to estimate the boundary of $(A^I)^k$ and use (12) to estimate the lower and upper boundaries of $h_k = CA^{k-1}B$.

V. ROBUST ILC DESIGN

At this point, we turn our discussion to iterative learning control assuming that we calculated $h^I_k \in [\underline{h_k}, \overline{h_k}]$ from A^I using above procedures. In what follows, the stability of interval ILC system is analyzed, and based on the stability analysis, ILC gains are designed to ensure monotonic convergence.The interval Markov matrix is expressed as:

$$H^I = \begin{bmatrix} h^I_1 & 0 & \cdots & 0 \\ h^I_2 & h^I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h^I_n & h^I_{n-1} & \cdots & h^I_1 \end{bmatrix}$$

where n is the total number of the discrete time points in time domain. Note that in this section, H^I is not a set, instead, it denotes interval matrices. For convenience, let H_0 be the nominal Markov matrix without interval uncertainty, calculated as:

$$H_0 = \left[\frac{\underline{h^I_i} + \overline{h^I_i}}{2}, i = 1, \cdots, n\right]$$

The following symbols T_0 and T^I are used to represent super-vector ILC: $T_0 := I_{n\times n} - H_0\Gamma$, and $T^I := I_{n\times n} - H^I\Gamma$, where $I_{n\times n}$ is an $n \times n$ identity matrix, and Γ is a learning gain matrix. For more detailed explanation of super-vector ILC, see [1], [3], [8].

A. Stability Analysis of Interval ILC

First, let us discuss the asymptotic stability of the interval ILC system, which is simply checked from the lower Toeplitz triangular structure of Markov matrix. For example, in $I - H^I\Gamma$, if Arimoto-like learning gains (denoted as γ) or purely causal gains are used, the diagonal terms are $1 - \gamma h_1$. When h_1 is an interval Markov parameter such as $h^I_1 \in [\underline{h_1}, \overline{h_1}]$, the asymptotic stability is checked from two extreme values of h^I_1. The following asymptotic stability condition was suggested in [11]: If $|1 - \gamma\underline{h_1}| < 1$ and $|1 - \gamma\overline{h_1}| < 1$, then the ILC system is asymptotically stable. Thus, based on this asymptotic stability condition, if Arimoto-like learning gains are designed such that $|1 - \gamma\underline{h_1}| < 1$ and $|1 - \gamma\overline{h_1}| < 1$, the interval ILC system will then be asymptotical stable. However, if the linear time-varying gains are used, the asymptotic stability condition is defined as: $\max\{|1 - \gamma_{ii}\underline{h_1}|, i = 1, \cdots, n\} < 1$ and $\max\{|1 - \gamma_{ii}\overline{h_1}|, i = 1, \cdots, n\} < 1$, where $\gamma_{ii}, i = 1, \cdots, n$ are time-varying diagonal gains in ILC learning gain matrix. If noncausal gains are used an existing interval matrix stability analysis method is utilized. Make the following substitutions:

$$\begin{aligned} s^1_{ij} &:= \overline{a_{ij}} \text{ if } i = j \\ s^1_{ij} &:= \max\{|\underline{a_{ij}}|, |\overline{a_{ij}}|\} \text{ if } i \ne j \\ s^2_{ij} &:= \underline{a_{ij}} \text{ if } i = j \\ s^2_{ij} &:= \min\{-|\underline{a_{ij}}|, -|\overline{a_{ij}}|\} \text{ if } i \ne j, \end{aligned}$$

where s^1_{ij} is i^{th} row and j^{th} column element of matrix S^1; s^2_{ij} is an element of matrix S^2, and a^I_{ij} is an element of general interval matrix A^I. Then the following lemma is adopted from the literature (for a proof, see Theorem 2 of [12]).

Lemma 5.1: Let an interval matrix be given in element wise as $\underline{A} \le A \le \overline{A},\ A \in A^I$. If $\beta = \max\{\rho(S^1), \rho(S^2)\} < 1$, where ρ is the spectral radius, then the interval matrix set A^I is Schur stable.

Next, let us consider monotonic convergence of the interval ILC problem. The monotonic convergence in 2-norm topology can be checked by Lemma 5.1 after small modifications. Here, however, consider the following error vector update law: $E_{k+1} =$

$(I - H^I\Gamma)E_k$, where the singular value of $(I - H^I\Gamma) = T^I$ is calculated as:

$$\overline{\sigma}\left(T^I\right) = \rho\left[(T^I)^T T^I\right] = \sqrt{\rho\begin{bmatrix} 0 & (T^I)^T \\ T^I & 0 \end{bmatrix}}$$

So, if the following interval matrix is defined:

$$\mathbf{H}^I = \begin{bmatrix} 0 & (T^I)^T \\ T^I & 0 \end{bmatrix}, \tag{13}$$

the monotonic convergence property of the interval ILC system is then checked by analyzing the Schur stability of $\mathbf{H}^I$ in the 2-norm topology using Lemma 5.1.

Monotonic convergence conditions in 1 and ∞ norm topologies are adopted from [11] as follows:

Lemma 5.2: Given $h_i^I \in [\underline{h_i}, \overline{h_i}]$, the interval ILC system is monotonically convergent, if

$$\|I - H^v\Gamma\|_k < 1, \tag{14}$$

where k is 1 or ∞; and H^v are vertex Markov matrices.

The main advantage of Lemma 5.2 is that the monotonix convergence property of the interval ILC system is checked just using vertex Markov matrices of H^I without using vertex matrices of T^I. This characteristic enables us to save the computational time and to reduce the conservativeness.

B. Design of Interval ILC

Let us assume that the interval uncertainties of the Markov parameters are calculated based on the perturbation method introduced in Section II and Section III. For achieving asymptotic stability, if Arimoto-like gains are selected such that $|1-\gamma\underline{h_1}| < 1$ and $|1 - \gamma\overline{h_1}| < 1$, the asymptotic stability of interval ILC system is achieved. To increase robustness, the following scheme is recommended:

$$\gamma = \begin{cases} 1/\overline{h_1} & \text{if} \quad h_1^I \geq 0 \\ 1/\underline{h_1} & \text{if} \quad h_1^I < 0 \end{cases} \tag{15}$$

For the monotonically convergent ILC gain design in 2-norm topology, we use (13). For the learning gain matrix design, the following optimization is suggested based on Lemma 5.1.

Suggestion 5.1: Let $\mathbf{h}_{ij}^I$ be the i^{th} row and j^{th} column element of $\mathbf{H}^I$. If we define a matrix $\mathbf{M}$ whose elements are given as:

$$\mathbf{m}_{ij} = \max\{\underline{\mathbf{h}_{ij}}, \overline{\mathbf{h}_{ij}}\} \tag{16}$$

then, we solve the following optimization problem to design Γ:

$$\min_{\Gamma} \rho(\mathbf{M}) \tag{17}$$

$$\text{s.t.} \quad h_k \in [|\underline{h_k}|, |\overline{h_k}|]$$

Remark 5.1: We explain why the matrix $\mathbf{M}$ is used in (17). Let us define $\mathbf{s}_{ij}^1$ ($\mathbf{S}^1 = \{\mathbf{s}_{ij}^1\}$) and $\mathbf{s}_{ij}^2$ ($\mathbf{S}^2 = \{\mathbf{s}_{ij}^2\}$) such as:

$$\begin{aligned} \mathbf{s}_{ij}^1 &:= \overline{\mathbf{h}_{ij}} \text{ if } i = j; \\ \mathbf{s}_{ij}^1 &:= \max\{|\underline{\mathbf{h}_{ij}}|, |\overline{\mathbf{h}_{ij}}|\} \text{ if } i \neq j; \\ \mathbf{s}_{ij}^2 &:= \underline{\mathbf{h}_{ij}} \text{ if } i = j; \\ \mathbf{s}_{ij}^2 &:= \min\{-|\underline{\mathbf{h}_{ij}}|, -|\overline{\mathbf{h}_{ij}}|\} \text{ if } i \neq j \end{aligned} \tag{18}$$

Then, since matrix $\mathbf{H}^I$ is symmetric, $\mathbf{S}^1 = -\mathbf{S}^2$; hence using the fact that $\rho(\mathbf{S}^1) = \rho(\mathbf{S}^2)$ and diagonal terms of $\mathbf{H}^I$ are all zeros, we only check the spectral radius of the matrix composed of the off-diagonal terms of $\mathbf{S}^1$, which is denoted as $\mathbf{M}$ in (16). Therefore, Suggestion 5.1 is reasonable.

If Lemma 5.2 is used, another optimization scheme is suggested without constraint such as:

Suggestion 5.2: If k is 1 or ∞, the following optimization is straightforward.

$$\min_{\Gamma} \|I - H^v\Gamma\|_k. \tag{19}$$

Several remarks follow.

Remark 5.2: When the time invariant Arimoto-like gains are used in Γ, the monotonic convergence of $\|E_k\|$ can be achieved if: $\min\{|h_1^I|\} > \sum_{i=2}^{i=n} \max\{|h_i^I|\}$, where Markov parameters h_i^I are intervals. This argument can be proved using the same method as performed in [4]. However, if $\min\{|h_1^I|\} \leq \sum_{i=2}^{i=n} \max\{|h_i^I|\}$, then as commented in [4], we may have to use a time-varying learning gain matrix, or fully or partially populated ILC gain matrix using causal and noncausal gains should be considered. Thus, when $\min\{|h_1^I|\} > \sum_{i=2}^{i=n} \max\{|h_i^I|\}$ is not satisfied, it is difficult to find an analytical solution for monotonic convergence.

Remark 5.3: The optimization is to minimize $\|I - H^v\Gamma\|_k$ using Γ, where Γ is a band-fixed learning gain matrix. In a small band size, it is possible that there could not exist optimization solution such that $\|I - H^v\Gamma\|_k < 1$. In this case, the band size should be increased until the optimization algorithm finds Γ such that $\|I - H^v\Gamma\|_k < 1$.

Remark 5.4: Depending on the interval ILC system, the optimization scheme suggested above may not find the optimization solution even with the fully populated learning gain matrix. In this case, the following control update law could be used:

$$U_{k+1} = Q(U_k + \Gamma E_k),$$

where Q is a time-invariant diagonal matrix. Then, since the error vector is updated by the following formula:

$$E_{k+1} = Q(I - H\Gamma)E_k + (I - Q)Y_d$$

it is easy to make $\|Q(I - H\Gamma)\| < 1$ by Q and Γ. However, the remaining term $(I - Q)Y_d$ makes a non-zero steady-state error. This is a trade-off.

VI. SIMULATION ILLUSTRATION

Let us assume that the following simple discrete servo system was given from a continuous system:

$$\begin{aligned} x_1(k+1) &= a_{11}x_1(k) + a_{12}x_2(k) + b_1u(k) \\ x_2(k+1) &= a_{21}x_1(k) + a_{22}x_2(k) + b_2u(k) \qquad (20) \\ y(k) &= c_1x_1(t) + c_2x_2(k), \qquad (21) \end{aligned}$$

where $b_1 = 2$, $b_2 = 0.5$, $c_1 = 1$, $c_2 = 0$, interval parameters are bounded as: $-0.74 \leq a_{11} \leq -0.66$, $-0.53 \leq a_{12} \leq -0.47$, $0.95 \leq a_{21} \leq 1.05$, and $0.19 \leq a_{22} \leq 0.21$, and u is the control force. After exchanging (20) and (21) into the state-space form, we find A_0 and ΔA^I such as:

$$A_0 = \begin{bmatrix} -0.7 & -0.5 \\ 1.0 & 0.2 \end{bmatrix}; \overline{\Delta A} = |\underline{\Delta A}| = \begin{bmatrix} 0.04 & 0.03 \\ 0.05 & 0.01 \end{bmatrix}$$

where $|\underline{\Delta A}|$ is the matrix composed of the absolute values of $\underline{\Delta A}$ element-wisely. Using the results of foregoing sections, a simulation test is performed with the following reference sinusoidal signal: $Y_d = \sin(8.0j/n)$, where $n = 20$ and $j = 1, \cdots, n$. The band size is fixed as 3. The left figure of Fig. 1 shows

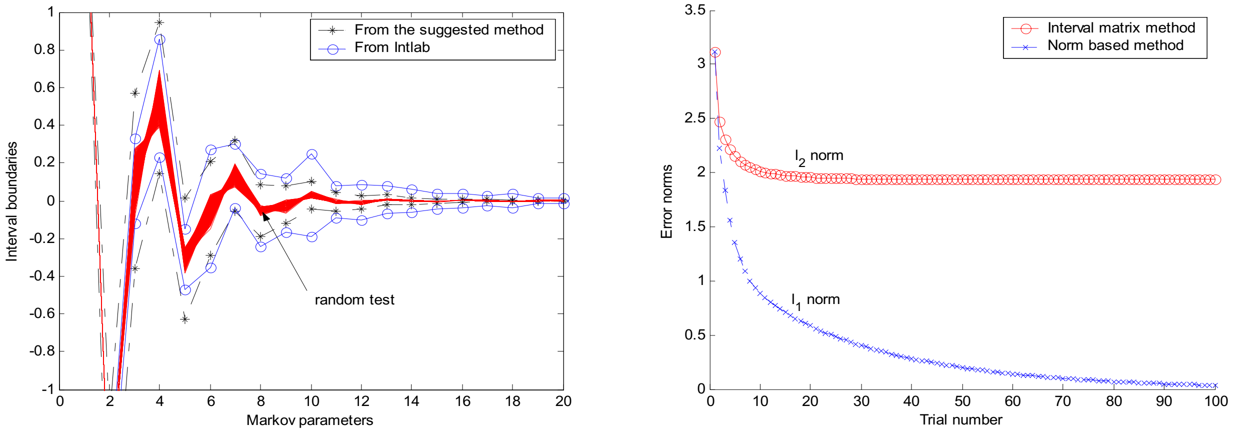

Fig. 1. Left: Calculated interval uncertain boundaries of Markov parameters. Right: ILC convergence test.

the calculated interval boundaries of the Markov parameters. The circle-marked line represents the maximum/minimum boundaries of the Markov parameters calculated from the $Intlab$ software [9]; and the $*$-marked line represents the Markov parameter boundaries calculated from the suggested method. For verification of the suggested method, a Monte-Carlo type random test was also performed as shown in figure (arrowed by "random test"). Observe that the suggested method gives reliable bounds of the interval ranges of the Markov parameters and less conservative than $Intlab$ after h_6 (but, from h_1 to h_5, $Intlab$ is slightly less conservative). So, it is recommendable to use the interval ranges of h_1 to h_5 calculated from $Intlab$ and use the interval ranges of h_6 to h_{20} calculated from the suggested method to reduce the conservativeness. In the right figure, the circle-marked line (termed as "interval matrix method") is the maximum l_2 norm error of the ILC, whose learning gain matrix was designed by (17); and the $\times$-marked line (termed as "norm-based method") is the maximum l_1 norm error, whose learning gain matrix was designed by (19). In the case of interval matrix method, there is a steady state error, because we fixed $Q = 0.9$ to guarantee the monotone convergence as commented in Remark 5.4) (when $Q = 1$, the optimization did not find the optimal solution such that the norm is less than 1).

VII. CONCLUDING REMARKS

In this paper, a robust iterative learning controller was designed with consideration of interval uncertainty of the system plant. The interval uncertainty of the system plant was converted into the super-vector iterative learning control system (i.e., into an interval Markov matrix). Optimization schemes were also suggested based on an interval matrix stability analysis method and a norm based method. In the case of the norm based method, the ILC learning gain matrix guaranteed the monotonic convergence of the uncertain ILC system with zero steady error. However, the interval matrix method only guaranteed the monotonic convergence with non-zero steady error. From these results, we conclude that the norm based method is less conservative than the interval matrix method. However, the norm based method requires much more computational time than the interval matrix method.

REFERENCES

[1] K. L. Moore, *Iterative learning control for deterministic systems*, Advances in Industrial Control. Springer-Verlag, 1993.

[2] K. L. Moore, "Iterative learning control - an expository overview," *Applied & Computational Controls, Signal Processing, and Circuits*, vol. 1, no. 1, pp. 151–241, 1999.

[3] Kevin L. Moore, "Multi-loop control approach to designing iterative learning controllers," in *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, Florida, USA, 1998, pp. 666–671.

[4] Kevin L. Moore, YangQuan Chen, and Vikas Bahl, "Monotonically convergent iterative learning control for linear discrete-time systems," *in Submitted to Automatica*, pp. 1–16, July 2003.

[5] K. L. Moore and YangQuan Chen, "On monotonic convergence of high order iterative learning update laws," in *Invited Session on* High-order Iterative Learning Control *at The 15-th IFAC Congress*, Barcelona, Spain, July 21-26 2002, IFAC.

[6] Kevin L. Moore and YangQuan Chen, "A separative high-order framework for monotonic convergent iterative learning controller design," in *Proceedings of 2003 American Control Conference*, Denver, Corolado, USA, June 2003, IEEE, pp. 3644–3649.

[7] K. L. Moore, "An iterative learning control algorithm for systems with measurement noise," in *Proc. of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona USA, Dec. 1999, IEEE, pp. 270–275.

[8] K. L. Moore, "On the relationhsip between iterative learning control and one-step ahead minimum prediction error control," in *Proc. of the Asian Control Conference*, Shanghai, Jul. 2000, pp. 1861–1865.

[9] G. I. Hargreaves, "Interval analysis in MATLAB," Numerical Analysis Report No. 416, The University of Manchester, December 2002.

[10] Leonard Meirovitch, *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, Rockville, Maryland, U.S.A., 1980.

[11] Hyo-Sung Ahn, Kevin L. Moore, and YangQuan Chen, "Stability analysis of iterative learning control system with interval uncertainty," in *submitted to 2005 IFAC World Congress*, 2005.

[12] J. J. D. Delgado-Romero, R. S. Gonzalez-Garza, J. A. Rojas-Estrada, G. Acosta-Villarreal, and F. Delgado-Romero, "Some very simple Hurwitz and Schur stability tests for interval matrices," in *Proceedings of the 30th Conference on Decision and Control*, Kobe, Japan, December 1996, IEEE, pp. 2980–2981.