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Brief paper

Monotonically convergent iterative learning control for linear discrete-time systems☆

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Abstract

In iterative learning control schemes for linear discrete time systems, conditions to guarantee the monotonic convergence of the tracking error norms are derived. By using the Markov parameters, it is shown in the time-domain that there exists a non-increasing function such that when the properly chosen constant learning gain is multiplied by this function, the convergence of the tracking error norms is monotonic, without resort to high-gain feedback.

Keywords: Iterative learning control; Discrete time system; Transient learning performance; Monotonic convergence

1. Introduction

Iterative learning control (ILC) has been proposed as a *value-added block* to enhance feedback controller performance when a system is operated repeatedly for the same task (Arimoto, Kawamura, & Miyazaki, 1984; Moore, 1993). While the formal mathematically rigorous analysis is initially due to Arimoto et al. (1984), the basic idea can be traced back to Uchiyama (1978) and even to Garden (1967), which is commented in Chen and Moore (2000). Detailed literature reviews and recent developments on ILC research can be found in Moore (1999), Bien and Xu (1998), Chen and Wen (1999), and Norrlöf (2000).

It was observed in Lee and Bien (1997) that, although the λ-norm of the tracking error[1] from iteration to iteration can be proved to decay monotonically, the ∞-norm or sup-norm may increase to a huge value before it converges to the desired level. This transient behavior, which is a serious concern in the practical application of ILC schemes, can be improved by using an exponentially decay learning gain as discussed in Lee and Bien (1997). One may argue that to make the convergence monotonic in the sup-norm or the 2-norm topology, one can use a high-gain feedback (Lucibello, 1993; Owens, 1992). However, in practice, this is not practical because the high-gain feedback may saturate the actuators.

The fact that in some ILC schemes the error can grow quite large before converging has also been qualitatively discussed in Jang and Longman (1994) and

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[1] The λ-norm of a continuous-time vector signal $f(t) \in R^n$, $t \in [0,T]$, is defined by $\|f(t)\|_\lambda \triangleq \max_{t\in[0,T]} e^{-\lambda t}\|f(t)\|_\infty$ where $\|f(t)\|_\infty \triangleq \max_{1\leqslant j\leqslant n}|f_j(t)|$ and $\lambda > 0$. It is noted that $\|f(t)\|_\lambda \leqslant \max_{t\in[0,T]}\|f(t)\|_\infty \leqslant e^{\lambda T}\|f(t)\|_\lambda$. For a discrete-time vector signal, the λ-norm can be defined similarly, but the exponential function may be any power function.

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Elci, Longman, Phan, Juang, and Ugoletti (1994) from a frequency-domain perspective. The effect can be explained as a result of the propagation of high-frequency components of the error by the ILC algorithm. Recently, in the time domain, a condition for monotonic convergence of the ∞-norm of tracking errors was established in Moore (2001). There are also some analysis results for monotonic convergence of ILC schemes via using approximate impulse response (Ishihara, Abe, & Takeda, 1992), via reduced sampling rate (Hillenbrand & Pandit, 2000), and for sampled data nonlinear systems (Tayebi & Zaremba, 1999). However, how to achieve the monotonic convergence for discrete time systems via a proper ILC updating law design has not been addressed in the literature to our best knowledge.

In this paper, similar to Moore (2001), we use the time-domain discussion, without resort to a high gain feedback. Rather, we seek to use a time-varying learning gain to achieve monotonic learning convergence. By using the Markov parameters, it is discovered that there exists a decreasing function such that when the properly chosen constant learning gain is multiplied by this function, the convergence of the 2-norm of tracking errors can be made monotonic. Our contribution differs from that of Lee and Bien (1997), where an exponentially decreasing learning gain is used, in that our discussions are in the discrete-time domain using the FIR model and the supervector notion, which makes the monotonic convergence analysis more straightforward.

The rest of this paper is organized as follows. In Section 2, the problem formulation and the classic learning convergence conditions are given. In Section 3, the convergence condition for a time-varying learning gain is established. Conditions to achieve monotonic convergence of tracking error norms are derived. Section 4 presents some illustrative examples. Finally, Section 5 concludes the paper.

2. Problem formulation and convergence conditions

Let an operation, or trial, of the system to be controlled be denoted by the subscript "k" and let time during a given trial be denoted by "t," where $t \in [0, N]$. Both t and k are integers. Each time the system operates the input to the system, $u_k(t)$, is stored, along with the resulting system error, $e_k(t) = y_d(t) - y_k(t)$, where $y_d(t)$ is the desired output. The plant to be controlled is a discrete-time, linear, time-invariant system of the form

$$\begin{aligned} Y(z) &= H(z)U(z) \\ &= (h_d z^{-d} + h_{d+1} z^{-(d+1)} + h_{d+2} z^{-(d+2)} + \cdots)U(z), \end{aligned} \tag{1}$$

where d is the relative degree of the system, z^{-1} is the standard delay operator in time, and the parameters h_i are the standard Markov parameters of the system $H(z)$. We will assume from here forward that $d = 1$. We will also assume the standard ILC reset condition: $y_k(0) = y_d(0) = y_0$ for all k. If we define the "supervectors" (Moore, 1998) $U_k = [u_k(0), u_k(1), \ldots, u_k(N-1)]^{\mathrm{T}}$, $Y_k = [y_k(1), y_k(2), \ldots, y_k(N)]^{\mathrm{T}}$, $Y_d = [y_d(1), y_d(2), \ldots, y_d(N)]^{\mathrm{T}}$, then the system can be written as

$$Y_k = HU_k, \tag{2}$$

where H is the matrix of Markov parameters of the plant, given by

$$H = \begin{bmatrix} h_1 & 0 & 0 & \ldots & 0 \\ h_2 & h_1 & 0 & \ldots & 0 \\ h_3 & h_2 & h_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_N & h_{N-1} & h_{N-2} & \ldots & h_1 \end{bmatrix}. \tag{3}$$

For this system, the learning controller's goal is to derive an optimal input $u^*(t)$, for $t \in [0, N-1]$ by evaluating the error $e_k(t) = y_d(t) - y_k(t)$ (equivalently, E_k, where $E_k = [e_k(1), e_k(2), \ldots, e_k(N)]^{\mathrm{T}}$.) on the interval $t \in [1, N]$. This is accomplished by adjusting the input from the current trial (U_k) to a new input (U_{k+1}) for the next trial. This adjustment is done according to an appropriate algorithm. In this paper, we are interested in the convergence properties of the so-called Arimoto-type discrete-time ILC algorithm:

$$u_{k+1}(t) = u_k(t) + \gamma e_k(t+1), \tag{4}$$

where γ is the constant learning gain.

The convergence properties of the Arimoto-type ILC algorithm have been well established in the literature. Using a contraction mapping approach it is easy to see that the ILC scheme converges if the induced operator norm satisfies

$$\|I - \gamma H\|_i < 1.$$

Note that this sufficient condition ensures monotone convergence in the sense of the relevant norm topology. It is also possible to give the following necessary and sufficient condition for convergence (Moore, 1998):

$$|1 - \gamma h_1| < 1. \tag{5}$$

Unfortunately, this second condition does not guarantee monotone convergence as shown in Moore (2001).

Given a learning gain γ that satisfies the necessary and sufficient condition for convergence (5), what are the conditions on the plant under which monotone convergence is also guaranteed for the same learning gain? This question is answered in Moore (2001) in the ∞-norm topology. That is, we consider the convergence of $\|e_k(t)\|_\infty \triangleq \max_{t \in [1,N]} |e_k(t)|$. In this case, a sufficient condition for monotone convergence in the ∞-norm topology is that

$$\begin{aligned} \|I - \gamma H\|_i &= \|I - \gamma H\|_1 \\ &= \max_j \sum_{i=1}^{N} |(I - \gamma H)_{ij}| \\ &= |1 - \gamma h_1| + |\gamma| \sum_{j=2}^{N} |h_j| < 1. \end{aligned} \tag{6}$$

In addition to the necessary and sufficient condition for convergence (5), the other conditions to guarantee the monotone convergence can be found in a theorem given in Moore (2001). An improved version of the theorem is given as follows:

Theorem 1. *For the system $Y_k = HU_k$ and the learning control algorithm $U_{k+1} = U_k + \gamma E_k$, if γ is chosen so that $|1 - \gamma h_1| < 1$, then $\|I - \gamma H\|_1 < 1$ if*

$$|h_1| > \sum_{j=2}^{N} |h_j|. \tag{7}$$

Proof. First, from (5), one can observe that γ and h_1 should have the same sign, i.e., $\gamma h_1 = |\gamma h_1| = |\gamma||h_1|$. Moreover, $\gamma h_1 \in (0, 2)$. We have three cases to discuss.

- *Case* 1: $\gamma h_1 \in (0, 1)$. From (6), we immediately get (7).
- *Case* 2: $\gamma h_1 \in (1, 2)$. From (6), (7) can be written as

  $$\|I - \gamma H\|_1 < \gamma h_1 - 1 + |\gamma| \sum_{j=2}^{N} |h_j| < 1. \tag{8}$$

  So, one gets

  $$\sum_{j=2}^{N} |h_j| < 2/|\gamma| - |h_1|. \tag{9}$$

  which gives (7) by using the fact that here $\gamma h_1 \in (1, 2)$, i.e., $2/|\gamma| < 2|h_1|$.
- *Case* 3: $\gamma h_1 = 1$. Obvious since $\|I - \gamma H\|_1 < |\gamma| \sum_{j=2}^{N} |h_j| < 1$, i.e., $\sum_{j=2}^{N} |h_j| < 1/|\gamma| = |h_1|$. □

Remark 2.1. Theorem 1 presents conditions for monotonic learning convergence of $\|e_k(t)\|_\infty$. Due the structure of H, it is easy to observe that

$$\max_j \sum_{i=1}^{N} |(I - \gamma H)_{ij}| = \max_i \sum_{j=1}^{N} |(I - \gamma H)_{ij}| = |1 - \gamma h_1| + |\gamma| \sum_{j=2}^{N} |h_j| \tag{10}$$

i.e., $\|I - \gamma H\|_1 = \|I - \gamma H\|_\infty$. Therefore, one can conclude that conditions in Theorem 1 also guarantee the monotonic convergence of $\|e_k(t)\|_2 \triangleq \sqrt{\sum_{t=1}^{N} |e_k(t)|^2}$.

Remark 2.2. In certain systems, for example, the system with high oscillatory impulse response, $\sum_{j=2}^{N} |h_j|$ can be very large. Even if we can choose $\gamma = 1/h_1$ to make $|1 - \gamma h_1| = 0$, the monotonic convergence of $\|e_k(t)\|_\infty$ and $\|e_k(t)\|_2$ may still be impossible. This is due to the fact that the monotonic convergence condition (7) is independent on the constant learning gain γ.

The major contribution of this paper is to use a time-varying learning gain $\lambda(t)$ to replace γ in the learning updating law (4). In what follows, we will show that, by properly choosing $\lambda(t)$, the monotonic convergence of the 1-norm and the 2-norm of the tracking error is achievable even if the condition (7) is not satisfied. It should be noted that it is a common measure in practice to use a root mean squares (RMS) value of the tracking error, which is proportional to $\|e_k(t)\|_2$.

3. Monotonic convergence by using a time-varying learning gain

Using a time-varying learning gain $\lambda(t)$, the learning updating law (4) becomes

$$u_{k+1}(t) = u_k(t) + \lambda(t) e_k(t + 1). \tag{11}$$

Let the varying learning gain $\lambda(t)$ be defined as follows:

$$\lambda(t) = \gamma \mathrm{e}^{-\alpha(t-1)}, \tag{12}$$

where α is a suitably chosen positive real number. Define the $N \times N$ matrix Γ by

$$\Gamma = \gamma \mathrm{diag}\{1, \mathrm{e}^{-\alpha}, \mathrm{e}^{-2\alpha}, \ldots, \mathrm{e}^{-(N-1)\alpha}\}.$$

First, we state the following necessary and sufficient condition for ILC convergence using the learning updating law (11):

Theorem 2. *For the system $Y_k = HU_k$ and the learning control algorithm $U_{k+1} = U_k + \Gamma E_k$, the learning process converges iff*

$$\rho_1 \triangleq |1 - \gamma h_1| < 1. \tag{13}$$

Proof. Straightforward by referring to Moore (1998) and using the fact that the condition $\rho_1 < 1$ implies that $\rho_j \triangleq |1 - \gamma h_1 \mathrm{e}^{-\alpha(j-1)}| < 1$, for $j = 2, 3, \ldots, N$. In particular, when $\gamma h_1 < 1$, $\rho_j < \rho_{j+1} < 1$, for all $j = 1, \ldots, N$. □

The above ILC convergence condition (5) cannot guarantee the monotonic convergence of $\|e_k(t)\|_2$ and $\|e_k(t)\|_1$. In this section, we will concentrate on the case when condition (7) in Theorem 1 is not satisfied. What we will show is that there exists a choice of α such that the monotonic convergence of $\|e_k(t)\|_1$ and $\|e_k(t)\|_2$ is achievable.

First, let $\bar{y}_k(t) = \mathrm{e}^{-\alpha(t-1)} y_k(t)$, $\bar{y}_d(t) = \mathrm{e}^{-\alpha(t-1)} y_d(t)$ and $\bar{e}_k(t) = \mathrm{e}^{-\alpha(t-1)} e_k(t)$. The corresponding "supervectors" are denoted by $\bar{Y}_k = [\bar{y}_k(1), \bar{y}_k(2), \ldots, \bar{y}_k(N)]^{\mathrm{T}}$, $\bar{Y}_d = [\bar{y}_d(1), \bar{y}_d(2), \ldots, \bar{y}_d(N)]^{\mathrm{T}}$, $\bar{E}_k = [\bar{e}_k(1), \bar{e}_k(2), \ldots, \bar{e}_k(N)]^{\mathrm{T}}$.

Then the transformed system can be written as

$$\bar{Y}_k = \bar{H} U_k, \tag{14}$$

where $\bar{H}$ is its matrix of Markov parameters given by

$$\bar{H} = \begin{bmatrix} h_1 & 0 & 0 & \dots & 0 \\ \mathrm{e}^{-\alpha}h_2 & \mathrm{e}^{-\alpha}h_1 & 0 & \dots & 0 \\ \mathrm{e}^{-2\alpha}h_3 & \mathrm{e}^{-2\alpha}h_2 & \mathrm{e}^{-2a}h_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-(N-1)\alpha}h_N & \mathrm{e}^{-(N-1)\alpha}h_{N-1} & \mathrm{e}^{-(N-1)\alpha}h_{N-2} & \dots & \mathrm{e}^{-(N-1)\alpha}h_1 \end{bmatrix} \qquad (15)$$

and (11) becomes

$$U_{k+1} = U_k + \Gamma E_k = U_k + \gamma \bar{E}_k. \qquad (16)$$

Simple manipulations yield

$$\bar{E}_{k+1} = (1 - \gamma \bar{H})\bar{E}_k. \qquad (17)$$

We will establish the monotonic convergence condition for $\|\bar{E}_k\|_\infty$ first. In addition to the convergence condition (13), a sufficient condition for monotone convergence of $\|\bar{E}_k\|_\infty$ is given in the following theorem.

Theorem 3. *For the system* $\bar{Y}_k = \bar{H}U_k$ *and the learning control algorithm* $U_{k+1} = U_k + \gamma\bar{E}_k$, *there exist a* γ *and an* $\alpha > 0$ *such that*

$$\sum_{j=2}^{N} \mathrm{e}^{-(j-1)\alpha}|h_j| < |h_1| \qquad (18)$$

and

$$\gamma h_1 \in (0, 1). \qquad (19)$$

Thus, *the monotonic convergence of* $\|\bar{E}_k\|_\infty$ *is guaranteed.*

Proof. To guarantee the monotonic convergence of $\|\bar{E}_k\|_\infty$, the following condition should be ensured:

$$\|I - \gamma\bar{H}\|_1 < 1. \qquad (20)$$

By the definition of the matrix 1-norm,

$$\begin{aligned} &\|I - \gamma\bar{H}\|_1 \\ &\quad = \max_i \left\{ |\gamma| \sum_{j=2}^{N-i+1} \mathrm{e}^{-(i+j-2)\alpha}|h_j| + |1 - \mathrm{e}^{-(i-1)\alpha}\gamma h_1| \right\} \\ &\quad = |\gamma| \sum_{j=2}^{N-i^*} \mathrm{e}^{-(i^*+j-2)\alpha}|h_j| + |1 - \mathrm{e}^{-(i^*-1)\alpha}\gamma h_1|, \qquad (21) \end{aligned}$$

where i^* is an integer between 1 and N. Again, it is noted that γ and h_1 should have the same sign according to (13). Due to (19), we know that $\mathrm{e}^{-i\alpha}\gamma h_1 < 1$ for all positive integer i. Therefore, (21) can be written as

$$\begin{aligned} &\|I - \gamma\bar{H}\|_1 \\ &\quad = |\gamma| \sum_{j=2}^{N-i^*+1} \mathrm{e}^{-(i^*+j-2)\alpha}|h_j| + 1 - \mathrm{e}^{-(i^*-1)\alpha}\gamma h_1 \\ &\quad = \mathrm{e}^{-(i^*-1)\alpha}|\gamma| \left(\sum_{j=2}^{N-i^*+1} \mathrm{e}^{-(j-1)\alpha}|h_j| - |h_1| \right) + 1. \qquad (22) \end{aligned}$$

To make the above less than 1, clearly, the following should be satisfied:

$$\sum_{j=2}^{N-i^*+1} \mathrm{e}^{-(j-1)\alpha}|h_j| < |h_1|. \qquad (23)$$

From the condition (18), the above is obvious because

$$|h_1| > \sum_{j=2}^{N} \mathrm{e}^{-(j-1)\alpha}|h_j| \geqslant \sum_{j=2}^{N-i^*+1} \mathrm{e}^{-(j-1)\alpha}|h_j|.$$

Therefore, for the system $\bar{Y}_k = \bar{H}U_k$ and the learning control algorithm $U_{k+1} = U_k + \gamma\bar{E}_k$, there exist a γ and an $\alpha > 0$ to achieve the monotonic convergence of $\|\bar{E}_k\|_\infty$. ☐

Remark 3.1. From Theorem 3 we know that it is always possible to make the convergence of $\|\bar{E}_k\|_\infty$ monotonic, i.e., $\|\bar{E}_{k+1}\|_\infty < \|\bar{E}_k\|_\infty$ for all k. Note that $\bar{e}_k(t) = \mathrm{e}^{-\alpha(t-1)}e_k(t)$. From the fact that $\max_{t\in[1,N]} |\bar{e}_{k+1}(t)| < \max_{t\in[1,N]} |\bar{e}_k(t)|$ for all k, one cannot conclude that $\max_{t\in[1,N]} \mathrm{e}^{\alpha(t-1)}|\bar{e}_{k+1}(t)| < \max_{t\in[1,N]} \mathrm{e}^{\alpha(t-1)}|\bar{e}_k(t)|$. Therefore, Theorem 3 does not guarantee the monotone convergence of $\|E_k\|_\infty$.Moreover, monotone convergence of $\|\bar{E}_k\|_\infty$ does not, in general, imply monotone convergence of $\|\bar{E}_k\|_1$ and $\|\bar{E}_k\|_2$.

However, in our case, we can show that there exists an α such that monotone convergence of $\|\bar{E}_k\|_1$ and $\|\bar{E}_k\|_2$ can be ensured. We need the following theorem.

Theorem 4. *There exists an* α *such that for all* k *and* t

$$|\bar{e}_{k+1}(t)| \leqslant |\bar{e}_k(t)|. \qquad (24)$$

Proof. The proof is in an enumerative way based on the error recursive equation (17). For $t = 1$, we have

$$|\bar{e}_{k+1}(1)| \leqslant \rho_1|\bar{e}_k(1)|.$$

Since from (13), $\rho_1 < 1$, $|\bar{e}_{k+1}(1)| \leqslant |\bar{e}_k(1)|$ for all k. For $t = 2$, from (17), we have

$$|\bar{e}_{k+1}(2)| \leqslant \rho_2|\bar{e}_k(2)| + \mathrm{e}^{-\alpha}|h_2||\bar{e}_k(1)|.$$

Note that $|\bar{e}_k(1)|$ is bounded and $|h_2|$ is finite. Also noted is that $|\bar{e}_k(1)|$ converges faster than $|\bar{e}_k(2)|$ since $\rho_1 < \rho_2 < 1$ (see the proof of Theorem 2). Therefore, clearly, there exists a large enough $\alpha = \alpha_1^*$ such that $|\bar{e}_{k+1}(2)| \leqslant |\bar{e}_k(2)|$ for all k. For $t = 3$, similarly,

$$|\bar{e}_{k+1}(3)| \leqslant \rho_3|\bar{e}_k(3)| + \mathrm{e}^{-2a}|h_3||\bar{e}_k(1)| + \mathrm{e}^{-2\alpha}|h_2||\bar{e}_k(2)|.$$

Obviously, there exist a large enough $\alpha = \alpha_2^* \geqslant \alpha_1^*$ such that $|\bar{e}_{k+1}(3)| \leqslant |\bar{e}_k(3)|$ for all k. Continuing the above enumerations ends the proof. $\square$

With Theorem 4, we can immediately conclude that there exists an α such that the convergence of $\|\bar{E}_k\|_1$ and $\|\bar{E}_k\|_2$ can be ensured to be monotonic, i.e.,

$$\sum_{t=1}^{N} |\bar{e}_{k+1}(t)| - \sum_{t=1}^{N} |\bar{e}_k(t)| \leqslant 0, \tag{25}$$

$$\sum_{t=1}^{N} |\bar{e}_{k+1}(t)|^2 \leqslant \sum_{t=1}^{N} |\bar{e}_k(t)|^2 \quad \text{and}$$
$$\sqrt{\sum_{t=1}^{N} |\bar{e}_{k+1}(t)|^2} - \sqrt{\sum_{t=1}^{N} |\bar{e}_k(t)|^2} \leqslant 0. \tag{26}$$

From the monotonicity of $\|\bar{E}_k\|_1$ and $\|\bar{E}_k\|_2$, as shown in (25) and (26), we can examine the monotonicity of $\|E_k\|_1$ and $\|E_k\|_2$. It is straightforward to see that

$$\begin{aligned}
&\|E_{k+1}\|_1 - \|E_k\|_1 \\
&\quad = \sum_{t=1}^{N} \mathrm{e}^{\alpha(t-1)} |\bar{e}_{k+1}(t)| - \sum_{t=1}^{N} \mathrm{e}^{\alpha(t-1)} |\bar{e}_k(t)| \\
&\quad = \sum_{t=1}^{N} \mathrm{e}^{\alpha(t-1)} (|\bar{e}_{k+1}(t)| - |\bar{e}_k(t)|) \leqslant 0,
\end{aligned} \tag{27}$$

$$\|E_{k+1}\|_2^2 - \|E_k\|_2^2 = \sum_{t=1}^{N} \mathrm{e}^{2\alpha(t-1)} (|\bar{e}_{k+1}(t)|^2 - |\bar{e}_k(t)|^2) \leqslant 0. \tag{28}$$

Thus, we can conclude that, by using a time-varying learning gain, it is always possible to achieve the monotonic convergence of $\|E_k\|_2$ and $\|E_k\|_1$.

Several remarks concerning our results are provided in the following:

Remark 3.2. The monotonic convergence condition (7) in Theorem 1 is a "passive" condition, i.e., there is no tuning knob for the designer. The basic intuition of this work is to provide a tuning knob to "actively" achieve the monotonic convergence. Introducing the time-varying learning gain, through the analysis in this section and simulation illustrations in the next section, it is shown that we can "actively" tune to equivalently achieve the "passive" condition (7) for the monotonic convergence of ILC. Mathematically speaking, with the restriction on the Toeplitz form of $I - \gamma H$ matrix, by tuning γ alone, it is harder to make the induced norm of $I - \gamma H$ less than 1. However, using $\bar{H}$, not in Toeplitz form, it is possible to tune the induced norm of $I - \gamma \bar{H}$ by both γ and α. A general form of time-varying learning gain $\gamma(t)$ can be used at the expense of too many tuning knobs (N) which will turn the idea presented in this paper impractical in real applications.

Remark 3.3. From the monotonic convergence conditions, we can observe that the smaller the time horizon N, the easier it is to achieve monotonic convergence for lightly damped or oscillatory systems. This was also illustrated via simulation in Moore (2001).

Remark 3.4. The time-varying learning gain $|\lambda(t)|$ can actually be any nonincreasing time function. It does not need to be continuous. So, in some practical applications, $|\lambda(t)|$ can also be scheduled as a staircase form, not necessary in the form of an exponential function, as experienced in our simulation studies. Due to the space limit, we will not include the relevant simulation results.

Remark 3.5 (*Design of* α). To systematically design the time-varying learning gain $|\lambda(t)|$, or simply the coefficient α in this paper, the algebraic framework presented in Hatonen, Moore, and Owens (2002) can be applied. In our case, we showed that there exist an α to ensure the monotonic convergence. How to design α depends on what knowledge we have about the system to be controlled. Here, we assume two scenarios: (1) an experimental impulse response ($h_i, i = 1, \ldots, N$) is available and (2) impulse responses from several experiments are available. In scenario-1, we apply a numerical optimization procedure to search for an optimal α and γ as follows:

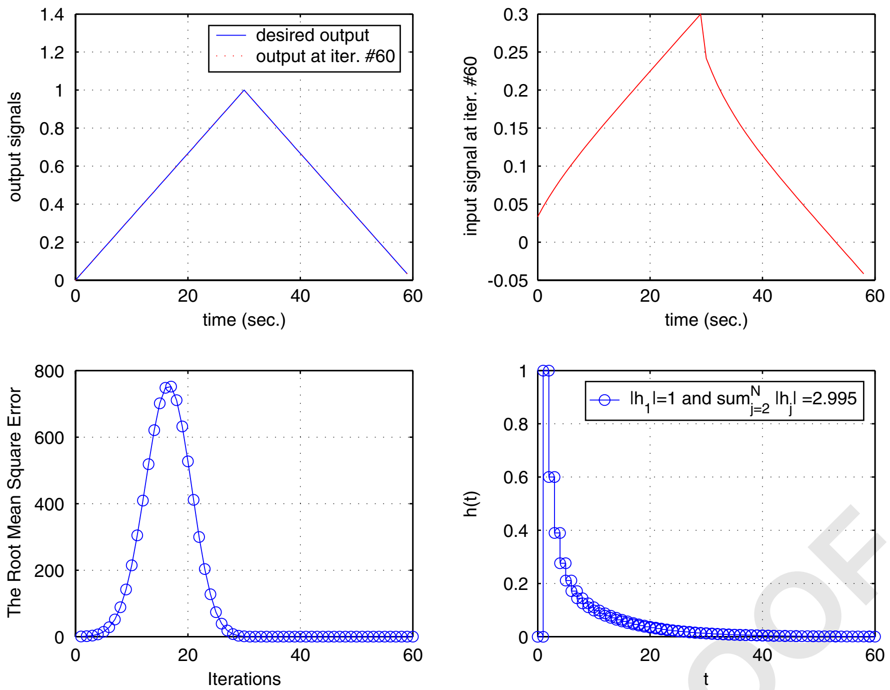
$$\min_{\alpha,\gamma} \|1 - \gamma \bar{H}\|_1.$$

Note that, when $\alpha = 0$, that is, when the scheme of this paper is not applied, it may happen that no γ can satisfy $\|1 - \gamma H\|_1 < 1$ while with α, this condition can be satisfied. In practice, the model uncertainty may exist and it is reasonable to assume that $h_i, i = 1, \ldots, N$ is in an interval, that is, $h_i \in [\underline{h}_i, \bar{h}_i]$ with the bounds $\underline{h}_i$ and $\bar{h}_i$ are known. Similarly, we can still apply the numerical optimization procedure to design an optimal α and γ using existing interval computation tools such as `IntLab` (Rump, 2004).

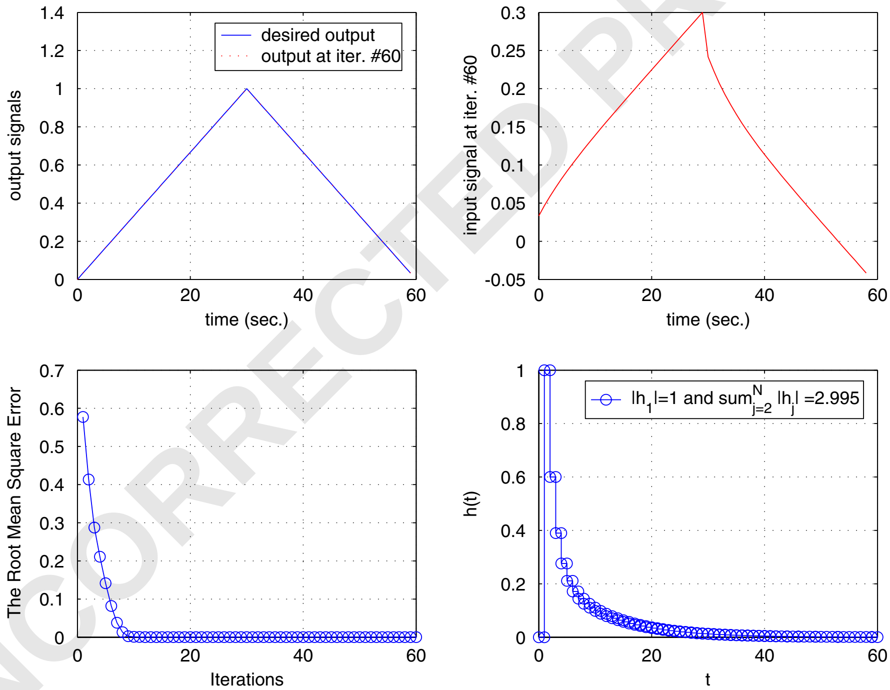
4. Illustrative examples

In this section, we shall present two examples to demonstrate the effectiveness of the proposed monotonic ILC scheme via using the time-varying learning gain. In all examples, we use a second-order IIR model for simulation. All initial conditions are set to 0. Further, h_1 is 1. Therefore, we fix $\gamma = 0.9$ such that $|1 - \gamma h_1| < 1$. In all simulations, we fix $N = 60$ and maximum number of iterations to 60. The desired trajectory is a triangle (ramp) with a maximum height 1 which is given by

$$y_d(t) = \begin{cases} 2t/N, & i = 1, \ldots, N/2, \\ 2(N - t)/N, & i = N/2 + 1, \ldots, N. \end{cases}$$



(a) $\alpha = 0$



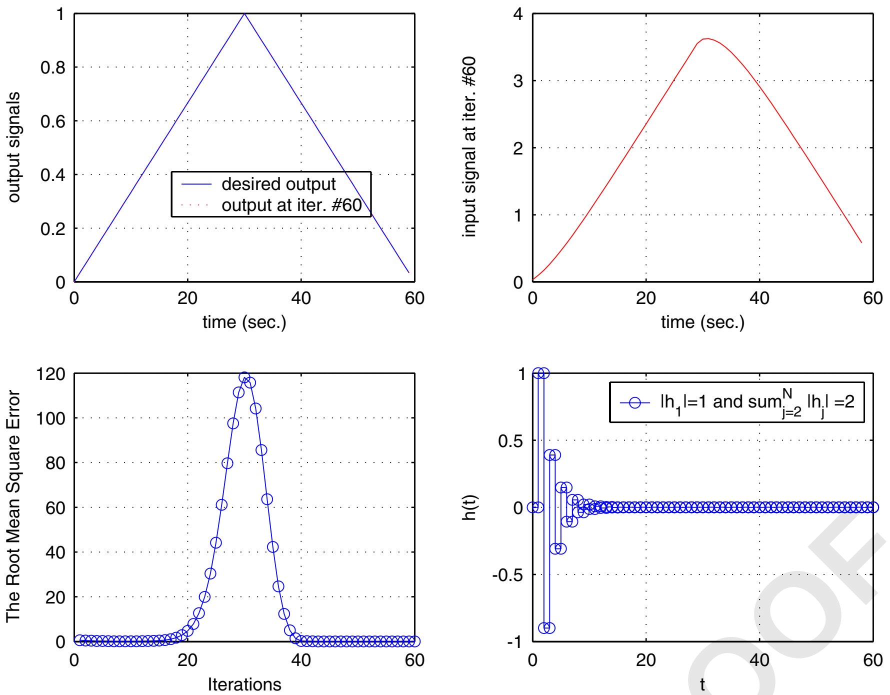
(b) $\alpha = 1.5/N$

Fig. 1. Stable lightly damped system $H_1(z)$: (a) $\alpha = 0$ and (b) $\alpha = 1.5/N$.

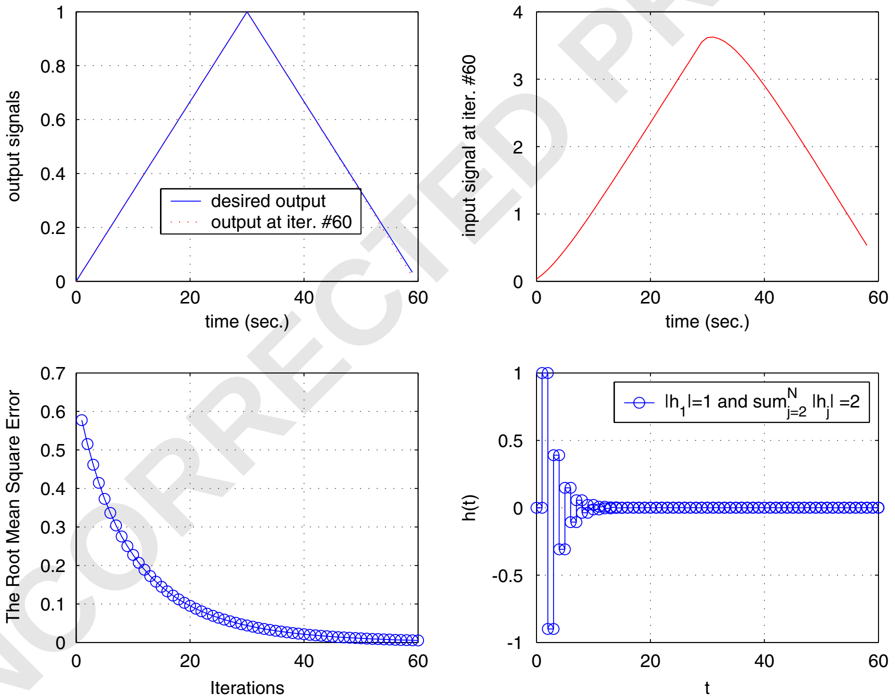
Case 1 (Stable lightly damped system): The z-transfer function for simulation is

$$H_1(z) = \frac{z - 0.8}{(z - 0.5)(z - 0.9)}.$$

Fig. 1(a) shows the ILC result using the constant learning gain $\gamma = 0.9$. This is equivalent to the proposed ILC scheme in (11) with $\alpha = 0$. The top left subplot of Fig. 1(a) shows the desired trajectory and the output at the 60th ILC iteration where we can observe the good tracking result. The desired input signal at the 60th ILC iteration is shown in the top right subplot of Fig. 1(a). The impulse response of $H_1(z)$ is shown in the bottom right subplot of Fig. 1(a) where we can read that $h_1 = 1$



(a) $\alpha = 0$



(b) $\alpha = 1.5/N$

Fig. 2. Stable oscillatory system $H_2(z)$: (a) $\alpha = 0$ and (b) $\alpha = 1.5/N$.

and $\sum_{j=2}^{60} = 2.995$. Clearly, the monotonic condition (6) is not satisfied. Therefore, in the bottom left subplot of Fig. 1(a), a quite big peak transient can be observed for the RMS of the tracking error with respect to iteration number.

Now, we apply the time-varying learning gain as described in the previous section. Choosing $\alpha = 1.5/N = 0.025$ in (11) gives the ILC result shown in Fig. 1(b). In this case, the ILC convergence is made monotonic as shown in the bottom left subplot of Fig. 1(b).

Case 2 (Stable oscillatory system): The z-transfer function for simulation in this case is

$$H_2(z) = \frac{z-0.8}{(z-0.5)(z+0.6)}.$$

Fig. 2(a) shows the ILC result using the constant learning gain $\gamma = 0.9$. The descriptions about the subplots are similar to Fig. 1(a). For $H_2(z)$, here $h_1 = 1$ and $\sum_{j=2}^{60} = 2$. Clearly, the monotonic condition (6) is not satisfied. Therefore, in the bottom left subplot of Fig. 2(a), a big peak transient can again be observed. With the same $\alpha = 1.5/N = 0.025$ as in Case 1, the ILC convergence is made monotonic as shown in the bottom left subplot of Fig. 2(b).

Remark 4.1. Note that for Cases 1 and 2, the same α works fine. This is due to the fact that the values of $\sum_{j=2}^{N} |h_j|$ in both cases are close. As a rule of thumb, the bigger the value of $\sum_{j=2}^{N} |h_j|$, the bigger the α. However, too big an α will slow down the convergence of the end portion of the trajectory. In practice, α should be regarded as a tuning knob to improve the possible bad or unsatisfactory transient performance in ILC.

5. Conclusion

The major discovery in this paper is that for linear discrete time systems, there exists a decreasing function such that when the properly chosen constant learning gain is multiplied by this function, the iterative learning convergence of the tracking error norms can be made monotonic, without resort to high gain feedback. Detailed derivations are presented together with some illustrative examples.

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