Proceedings of the 2006 American Control Conference
Minneapolis, Minnesota, USA, June 14-16, 2006

WeA07.1

On Initial Conditions in Iterative Learning Control

Jian-Xin Xu , Rui Yan and YangQuan Chen

Abstract— **Initial conditions, or initial resetting conditions, play a fundamental role in all kinds of iterative learning control methods. In this work we study five different initial conditions, disclose the inherent relationship between each initial condition and corresponding learning convergence (or boundedness) property. The iterative learning control method under consideration is based on Lyapunov theory, which is suitable for plants with time varying parametric uncertainties and local Lipschitz nonlinearities.**

I. INTRODUCTION

Learning control enhances the system performance through repeated or cyclic operations. Iterative learning control deals with finite time interval tracking tasks that repeat, whereas repetitive learning control copes with periodic tracking tasks over infinite time interval.

To make a process convergent in a finite time interval, the initial condition becomes crucial because asymptotical convergence along the time horizon is no longer valid. Iterative learning control (ILC) based on contraction mapping requires the identical initial condition ($i.i.c.$) in order to achieve a perfect tracking [1-4]. The robustness of contraction based ILC has been studied [5-10] and several algorithms were proposed for ILC without $i.i.c.$ [11-13].

Recently, new ILC approaches based on Lyapunov theory [4,14-17] have been developed to complement the contraction mapping based ILC in the sense that local Lipschitz nonlinearities can be taken into consideration. Majority of those approaches also require the identical initial condition. In practical applications, the perfect initial resetting may not be obtainable. That motivates us to study initial conditions for this class of ILC.

In the paper, five different initial conditions to be investigated are: a) identical initial condition ($i.i.c.$); b) progressive $i.i.c.$, i.e. the sequence of initial errors belong to l^2; c) fixed initial shift; d) random initial condition within a bound; e) alignment condition, i.e., the end state of the preceding iteration becomes the initial state of the current iteration.

Condition b) has not been exploited in contraction mapping based ILC. In the Lyapunov based ILC, this condition has been briefly mentioned in [18] wherein the unknowns are constant parameters. Hence, analogous to adaptive control, differential type adaptation law can be derived by the use of a quadratic Lyapunov function. In this work, we consider more general time-varying parametric uncertainties, wherein a difference type learning law is derived from a Lyapunov functional. A contribution of this work is to show the pointwise learning convergence under Condition b).

Condition c) has been studied in contraction mapping based ILC. In [11], it shows that the tracking error can converge exponentially along the time axis from the fixed initial shift which cannot be eliminated. In [12], by rectifying the reference trajectory nearby the initial stage into a new one aligned with the actual initial value, the uniform convergence of the tracking error can be achieved. Condition c) has not been studied in Lyapunov based ILC. A contribution of this work is to demonstrate the similar learning performance: the tracking error will enter a designated bound with the fixed initial shift, and pointwisely converges when the reference trajectory can be rectified.

The effect of Condition d), which reflects the ILC robustness property, has been investigated in contraction mapping based ILC, e.g. [9] and [11]. The results show that the tracking error is confined to a bound which depends continuously on the bound of the initial state error. In a special case of Condition d), an initial state learning algorithm [13] has been proposed to make the initial state a convergent sequence, subject to the maneuverability of the system initial states. By a rectifying action [12], the tracking error can also be confined to a finite bound which is proportional to the bound of the initial state error. As for the Lyapunov based ILC, the only report on Condition d) was given by [16], in which a switching control together with a reducing deadzone is used. In comparison, the contribution in this work is to show that the proposed ILC, which is a continuous control law, can converge to a designated bound under Condition d), or converge pointwisely when an appropriate rectifying action is taken.

Condition e) is not applicable in contraction mapping based ILC. In Lyapunov based ILC, our previous work [19] has shown the learning convergence under Condition e). In this work, we first show that the learning convergence or boundedness with respect to conditions a-d) and e), though very different, can be easily discussed and determined under a unified framework using a Lyapunov functional. Next, under the same framework, the learning convergence speed can be evaluated for the conditions c), d) and e).

The objective of ILC is to achieve a convergent sequence in a function space. As such, the sequence approaches the desired one either in a pointwise manner, in L^p norm or in uniform norm. In the analysis of contraction based ILC, often the uniform norm is used. However, the uniform convergence is rather difficult to achieve in many control problems, especially for tracking tasks in a function space. In this work,

Jian-Xin Xu and Rui Yan are with the department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117576. Tel: (+65)-68742566. E-mail: elexujx@nus.edu.sg.

YangQuan Chen is with Dept. of Electrical and Computer Engineering, Utah State University, 4120 Old Main Hill, Logan, UT 84322-4120, USA. E-mail: yqchen@ece.usu.edu.

we demonstrate that, a learning sequence can converge either pointwisely or in L^2 norm.

The paper is organized as follows. Section II states the problem and ILC algorithm. In Section III, the learning convergence properties are analyzed under different initial conditions. Section IV presents an illustrative example.

In the paper L^2 norm is defined as $\|e_i\|_T \triangleq (\int_0^T e_i^2 dt)^{\frac{1}{2}}$.

II. Problem Statement

Considering a tracking task that ends in a finite interval and repeats, ILC applies from iteration to iteration. To focus on the main theme with initial conditions, consider simple first order nonlinear dynamic system in the $i-$th iteration

$$\dot{x}_i = \theta(t)\xi(x_i, t) + u_i \qquad x(0) = x_0, \tag{1}$$

where $\xi(x_i, t)$ is a known nonlinear function which can be local Lipschitzian and the unknown time-varying parameter $\theta(t) \in \mathcal{C}[0,\ T]$. For notational convenience, in subsequent context we will omit the argument t for all variables and denote a function $\xi(x_i, t)$ as ξ_i where no confusion arises.

The reference trajectory is generated by a dynamics

$$\dot{x}_r = f(x_r, r, t), \tag{2}$$

where $f_r = f(x_r, r, t)$ is a known smooth function, r is a reference input which yields a bounded state $x_r(t)$ over the interval $[0,\ T]$. The tracking error is defined as $e_i(t) = x_r(t) - x_i(t)$.

The objective of ILC is to find a sequence of appropriate control input $u_i(t)$ for $t \in [0,\ T]$ such that the system state x_i tracks the reference trajectory x_r as $i \to \infty$.

From the theory of differential equation, the orbit of the nonlinear dynamics (1) is jointly determined by the initial value x_0 and the exogenous input u_i. A tiny discrepancy in initial conditions may lead to completely different orbits. However, a perfect initial resetting requires that the control system be equipped with a precise homing mechanism, which may not be possible for many practical engineering systems. Henceforth, the ultimate objective of this paper is to relax this requirement with several less strict initial conditions, and investigate how does the learning performance alter accordingly. Consider the following five initial conditions:

a) $e_i(0) = 0$;
b) $\sum_{i=1}^{\infty} e_i^2(0) = C$, where C is a constant;
c) $|e_i(0)| = C \neq 0$, where C is a constant;
d) $e_i(0)$ is random and bounded by a constant C;
e) $e_i(0) = e_{i-1}(T)$.

Condition a) is the identical initial condition ($i.i.c.$) that is widely assumed for most ILC algorithms. Condition b) is the progressive $i.i.c.$, it shows that the sequence of $\{e_i(0)\}$ belongs to l^2, or $e_i(0) \to 0$ as $i \to \infty$. Condition c) is the fixed initial shift. Obviously, Condition a) is a special case of Condition b), and Conditions a-c) are special cases of Condition d). Generally speaking, it is adequate to consider Condition d) the worst case, if our concern is regarding the ILC robustness on initial shifts. Nonetheless, we can derive better and quantitative results on learning convergence with Conditions a-d), as we will show in this work.

Condition e) is the alignment condition, which is different from other initial conditions. The initial resetting condition in ILC usually implies both spatial resetting and temporal resetting. While time resetting is natural for a task to be finished and repeated over a finite period, the spatial resetting is however not an easy job and not so imperative. Note that it is the spatial resetting which gives rise to extra implementation difficulty. In quite a number of practical applications, the process will restart from where it stopped in previous trial. Therefore the end state of the preceding iteration becomes the initial state of the new iteration, i.e. $x_{i-1}(T) = x_i(0)$. As far as the reference trajectory is spatially closed, namely $x_r(0) = x_r(T)$, Condition e) holds for all iterations. The alignment condition removes the spatial resetting requirement.

The error dynamics at the i-th iteration can be expressed as

$$\dot{e}_i = f_r - \theta(t)\xi_i - u_i. \tag{3}$$

The learning control mechanism consists of the control law

$$u_i = ke_i + f_r - \hat{\theta}_i(t)\xi_i, \tag{4}$$

and the parametric learning law

$$\hat{\theta}_i(t) = \text{proj}(\hat{\theta}_{i-1}(t)) - \xi_i e_i(t) \qquad \hat{\theta}_{-1}(t) = 0, \tag{5}$$

where

$$\text{proj}(\cdot) \triangleq \begin{cases} \cdot & |\cdot| \le \theta^* \\ \text{sign}(\cdot)\theta^* & |\cdot| > \theta^* \end{cases}$$

and θ^* is the projection bound which is sufficiently large such that $\theta^* \ge \sup_{t\in[0,\ T]} |\theta(t)|$. In practice, θ^* can be arbitrarily large but finite.

Substituting the learning control law (4) into the error dynamics (3) yields the closed-loop error dynamics

$$\dot{e}_i = -ke_i - \phi_i(t)\xi_i, \tag{6}$$

where $\phi_i(t) \triangleq \theta(t) - \hat{\theta}_i(t)$.

III. Learning Convergence Under Initial Conditions

First derive the boundedness of tracking error e_i and parameter estimate $\hat{\theta}_i$ under learning control law (4) and (5). Note that at the initial iteration $i = 0$, there is no parametric learning as $\hat{\theta}_{-1}(t) = 0$, and $\hat{\theta}_0 = -\xi_0 e_0(t)$. Hence we have to derive the boundedness of $(e_0, \hat{\theta}_0)$ in a way different from that for $(e_i, \hat{\theta}_i)$ with $i \ge 1$.

Proposition 1: $(e_0, \hat{\theta}_0)$ is bounded for $t \in [0,\ T]$.

Proof is given in Appendix A.

Now we can prove the boundedness of $(e_i, \hat{\theta}_i)$, which is summarized in the following theorem.

Theorem 1: Under the initial conditions a)-d), the learning control law (4) and (5) ensures bounded $(e_i, \hat{\theta}_i)$ for any $i \ge 1$.

Proof is given in Appendix B.

Since any two iterations are correlated via the learning law, the impact from an initial condition to the system performance could be in an accumulative fashion. The following proposition describes such an accumulative impact and facilitate subsequent analysis on the relationship between initial conditions and learning convergence.

Proposition 2: The inequality

$$\begin{aligned}\lim_{i\to\infty} V_i(t) &\leq V_0(t) + \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i} e_j^2(0) \\ &\quad - \lim_{i\to\infty}\sum_{j=1}^{i}\int_0^t k e_j^2 d\tau - \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i-1} e_j^2(t)\end{aligned} \tag{7}$$

holds for $\forall\ i$, where V_i is a Lyapunov functional defined as $V_i(t) = \frac{1}{2}e_i^2(t) + \frac{1}{2}\int_0^t \phi_i^2(\tau)d\tau$.

Proof is given in Appendix C.

Now we are in a position to demonstrate the main results summarized in theorem 2. First, in addition to the boundedness of $(e_i, \hat{\theta}_i)$, we can achieve better learning performance under initial conditions a-d). Second, we are able to achieve L^2 learning convergence under the alignment condition e). Third, under the same framework with the Lyapunov functional, it is possible to further evaluate the learning convergence speed.

Theorem 2: Part 1. Under the initial conditions a) and b), the tracking error e_i converges to zero pointwisely as $i \to \infty$;
Part 2. Under the initial conditions c) and d), there exists a subsequence $\{e_{i_j}\}$ of $\{e_i\}$ such that for any arbitrary $\delta > 0$, $\|e_{i_j}\|_T \leq \epsilon$ as $i_j \to \infty$, where $\epsilon = \sqrt{\frac{C^2+\delta}{2k}}$.
Part 3. Under the alignment condition e), the tracking error $\|e_i\|_T$ converges to zero as $i \to \infty$.
Part 4. Under the conditions c) and d), for any given $\epsilon_0 > 0$ and $k > \frac{C^2}{2\epsilon_0^2}$, the tracking error $\|e_i\|_T$ will enter the ϵ_0-bound after at most $\frac{2V_0(T)}{2k\epsilon_0^2 - C^2}$ iterations. Furthermore, under the condition e), the tracking error $\|e_i\|_T \leq \epsilon_0$ after at most $\frac{2V_0(T)+e_1^2(0)}{2k\epsilon_0^2}$ iterations.

Proof: Part 1

First consider the initial condition a). With the condition, (7) is $\lim_{i\to\infty} V_i(t) \leq V_0(t) - \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i-1} e_j^2(t)$. Consider the positiveness of V_i and boundedness of V_0, the sequence $e_i(t)$ converges to zero pointwisely as $i \to \infty$.

Next consider the initial condition b), $\sum_{i=1}^{\infty} e_i^2(0) = C$. The relation (7) becomes $\lim_{i\to\infty} V_i(t) \leq V_0(t) + \frac{1}{2}C - \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i-1} e_j^2(t)$. The convergence property is analogous to a) because C is finite.

Part 2

The reduction to absurdity is applied. Suppose, on the contrary, there exists a positive integer N such that $\|e_i\|_T \geq \epsilon$ for all $i \geq N$.

Let $t = T$. The relation (7) with the initial conditions c) and d), $|e_i(0)| \leq C$, is

$$\begin{aligned}\lim_{i\to\infty} V_i(T) &\leq V_0(T) + \lim_{i\to\infty}\frac{1}{2}iC^2 - \lim_{i\to\infty}\sum_{j=1}^{i}\int_0^T k e_j^2 d\tau \\ &\quad - \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i-1} e_j^2(T) \\ &\leq V_0(T) + \frac{1}{2}NC^2 - \sum_{j=1}^{N}\int_0^T k e_j^2 d\tau \\ &\quad + \lim_{i\to\infty}\frac{1}{2}(i-N)C^2 - \lim_{i\to\infty}\sum_{j=N}^{i}\int_0^T k e_j^2 d\tau \\ &\leq B + \lim_{i\to\infty}\frac{1}{2}(i-N)C^2 - \lim_{i\to\infty}(i-N)k\epsilon^2 \\ &= B + \lim_{i\to\infty}(i-N)(\frac{1}{2}C^2 - k\epsilon^2)\end{aligned} \tag{8}$$

where $B = V_0(T) + \frac{1}{2}NC^2 - \sum_{j=1}^{N}\int_0^T k e_j^2 d\tau$ is a finite constant. For arbitrary $\delta > 0$ and $\epsilon = \sqrt{\frac{C^2+\delta}{2k}}$, substitution into (8) we can obtain $\lim_{i\to\infty} V_i(T) \leq B - \lim_{i\to\infty}\frac{1}{2}(i-N)\delta$. The right hand side approaches $-\infty$ since B is finite, which leads to a contradiction with the fact that $V_i(T) \geq 0$.

Part 3

Let $t = T$ in (7). With the alignment condition e), $e_i(0) = e_{i-1}(T)$, we obtain the following relationship $\frac{1}{2}\sum_{j=1}^{i} e_j^2(0) - \frac{1}{2}\sum_{j=1}^{i-1} e_j^2(T) = \frac{1}{2}e_1^2(0)$, and

$$\lim_{i\to\infty} V_i(T) \leq V_0(T) + \frac{1}{2}e_1^2(0) - \lim_{i\to\infty}\sum_{j=1}^{i}\int_0^T k e_j^2 d\tau.$$

Therefore $\lim_{i\to\infty}\int_0^T e_i^2 dt \triangleq \lim_{i\to\infty}\|e_i\|_T^2 = 0$ because of the positiveness of V_i and the boundedness of V_0.

Part 4

Under the initial conditions c) and d), from (8) we have

$$\begin{aligned}V_i(T) &\leq V_0(T) + \frac{1}{2}iC^2 - \sum_{j=1}^{i}\int_0^T k e_j^2 d\tau \\ &= V_0(T) + \frac{1}{2}iC^2 - k\sum_{j=1}^{i}\|e_j\|_T^2.\end{aligned} \tag{9}$$

From (9), the larger the $\|e_j\|_T$, the faster the decease of $V_i(T)$. Let us assume a slowest decrease in $V_i(T)$, which corresponds to $\|e_j\|_T = \epsilon_0$ for all $j = 1, 2, \cdots, i$.
Since $0 \leq V_0(T) + \frac{1}{2}iC^2 - k\sum_{j=1}^{i}\|e_j\|_T^2$, substituting $\|e_j\|_T = \epsilon_0$, we can derive $i \leq \frac{2V_0(T)}{2k\epsilon_0^2 - C^2}$ and $k > \frac{C^2}{2\epsilon_0^2}$.

Under the initial condition e), by observing the inequality

$$V_i(T) \leq V_0(T) + \frac{1}{2}e_1^2(0) - k\sum_{j=1}^{i}\|e_j\|_T^2,$$

the larger the $\|e_j\|_T$, the faster the decrease of $V_i(T)$. Similarly, substituting $\|e_j\|_T = \epsilon_0$ into the inequality $0 \leq V_0(T) + \frac{1}{2}e_1^2(0) - k\sum_{j=1}^{i}\|e_j\|_T^2$, we can obtain $i \leq \frac{2V_0(T)+e_1^2(0)}{2k\epsilon_0^2}$. ■

Note that, in the Lyapunov based ILC, the state variables are accessible. A rectifying action can be taken to revise the reference trajectory such that its initial values are aligned with the actual ones. This leads to an improved learning performance for the initial conditions c) and d), as stated by the following corollary.

Corollary 1: Let revised reference trajectory x_r^* be

$$x_r^* = \begin{cases} x_r & \text{if} \quad t \in [h, T], \\ \tilde{x}_r & \text{if} \quad t \in [0, h), \end{cases} \tag{10}$$

where $h \in [0, T]$ can be chosen arbitrary and $\tilde{x}_r$ is a smooth function to link the initial position $x_i(0)$ and the reference trajectory $x_r(h)$ at the moment $t = h$. The less the h, the closer the revised reference trajectory to the original reference trajectory.

Obviously, $e_i(0) = 0$, i.e., initial condition a) is satisfied for the new reference trajectory. An interesting observation is, the tracking error dynamics (6) remains the same with respect to the new reference trajectory, even though the reference trajectory may vary at every iteration. Therefore, the pointwise convergence can be directly achieved in analogy to the result of initial condition a) in Theorem 2.

Remark 1: From Part 3 of Theorem 2, a large gain k can reduce the tracking error bound ϵ under the initial conditions c) and d). From Part 4 of Theorem 2, it can be seen that a large feedback gain k can also expedite the learning convergence speed.

Remark 2: To speed up the parametric learning, a learning gain $\gamma > 0$ can be introduced in the parametric learning law

$$\hat{\theta}_i = \hat{\theta}_{i-1} - \gamma \xi_i e_i.$$

Accordingly a factor γ^{-1} shall be multiplied to integral terms on the right hand side of Lyapunov functional, and the convergence analysis remains the same.

Remark 3: It should be noted that in deriving the above convergence properties, we consider only sufficient conditions or the worst case performance. In practice, we may achieve better learning performance such as uniform convergence, although in theory only pointwise or L^2 convergence is guaranteed.

IV. Illustrative Example

Consider the system

$$\dot{x} = (1 + \sin \pi t)x^2 + u \qquad x(0) = x_0.$$

The reference model is $\dot{x}_r = -x_r + \sin^2 \pi t + 2$ with $x_r(0) = 1$. The tracking interval is $[0,\ 2]$. Throughout the simulation, choose the feedback gain $k = 1$ and parametric learning gain $\gamma = 1$. To measure the performance, we either calculate the sup-norm $|e_i|_{sup}$, i.e., the maximum tracking error of $|e_i(t)|$ over $[0,\ 2]$, or calculate L^2 norm $\| \cdot \|_{T=2}$.

Due to the page limit, we only consider conditions b), d) and e).

Initial Condition b)

Let $e_i(0) = \frac{1}{i+1}$, then $C = \sum_{i=1}^{\infty} e_i^2(0) = (\frac{\pi^2}{8} + \frac{\pi^2}{6}) - 2$ is finite. The sup-norm of tracking error is displayed in Fig.1. It can be seen that the tracking error does converge, but not as fast as Condition a) due to the initial perturbations.

Initial Condition d)

Let $e_i(0)$ take values randomly in $[-0.3,\ 0]$. The tracking error convergence is given in Fig.2. It can be seen that, despite the large initial error, the tracking error is kept at a much lower level for most time.

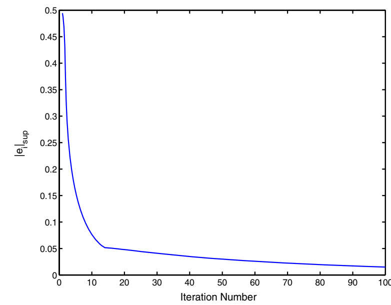

Fig. 1. Learning convergence under initial condition b).

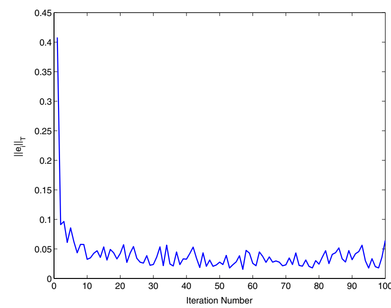

Fig. 2. Learning convergence under initial condition d)

According to corollary 1, the pointwise convergence of tracking error can be achieved if taking a rectifying action. In this example, for each iteration i, the reference trajectory is revised as the following

$$x_{r,i}^* = \begin{cases} x_r, & \text{if} \quad t \in [h, T] \\ A_i t^2 + B_i t + C_i, & \text{if} \quad t \in [0, h) \end{cases}$$

where

$$\begin{aligned} A_i &= \frac{\dot{x}_r(h)h + x_i(0) - x_r(h)}{h}, \\ B_i &= -\frac{\dot{x}_r(h)h + 2x_i(0) - 2x_r(h)}{h}, \\ C_i &= x_i(0). \end{aligned}$$

Clearly, the revised reference trajectory remains the same in the time interval $[h, T]$. The coefficients of the quadratic function are chosen such that the revised portion $x_{r,i}^*(t)$ and its derivative are aligned with the original reference trajectory at $t = h$, meanwhile the revised reference trajectory is aligned with the initial state value at $t = 0$. Choose $h = 0.3$, the pointwise convergence of the tracking error is shown in Fig.3.

Initial Condition e)

Finally consider a spatially closed reference $x_r(t) = 1 - \cos(\pi t)$, i.e. $x_r(0) = x_r(2)$. Theoretically, in this case the tracking error only converges according to $\| \cdot \|_T$. Let $k = 3$ and $\gamma = 5$. The tracking error according to $\| \cdot \|_T$ norm is displayed in Fig.4. It validates the learning effect.

V. Conclusion

We discussed five different initial conditions associated with ILC. For each initial condition, the boundedness along

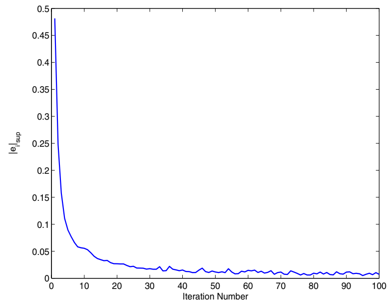

Fig. 3. Pointwise convergence under initial condition d) by rectifying the reference trajectory

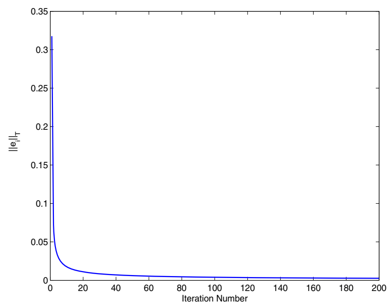

Fig. 4. Learning convergence under initial condition e)

the time horizon and asymptotical convergence along the iteration axis were exploited with rigorous analysis. Through both theoretical study and numerical examples, we can conclude that, the Lyapunov based ILC can effectively work with sufficient robustness.

Appendix A: Proof of Proposition 1

Proof: Choose Lyapunov functional

$$V_0(t) = \frac{1}{2}e_0^2(t) + \frac{1}{2}\int_0^t \phi_0^2(\tau)d\tau. \quad (11)$$

The upper right hand derivative of V_0 is

$$\dot{V}_0 = e_0\dot{e}_0 + \frac{1}{2}\phi_0^2 = -ke_0^2 - \phi_0\xi_0 e_0 + \frac{1}{2}\phi_0^2.$$

Noticing that $\hat{\theta}_0 = -\xi_0 e_0$, $\dot{V}_0$ becomes

$$\dot{V}_0 = -ke_0^2 + \phi_0\hat{\theta}_0 + \frac{1}{2}\phi_0^2 = -ke_0^2 - \frac{1}{2}\phi_0^2 + \phi_0\theta.$$

Using Young's inequality, for any $c > 0$ we have $\phi_0\theta \le c\phi_0^2 + \frac{1}{4c}\theta^2$. Let $0 < c < \frac{1}{2}$, $\dot{V}_0 \le -ke_0^2 - (\frac{1}{2} - c)\phi_0^2 + \frac{1}{4c}\theta^2$. Since $\theta(t) \in \mathcal{C}[0,\ T]$, there exists a finite bound $\theta_m \ge \theta(t)$ for any $t \in [0,\ T]$. Thus $\dot{V}_0$ is negative definite outside the region $\{(e_0, \phi_0) \in \mathcal{D} \mid ke_0^2 + (\frac{1}{2} - c)\phi_0^2 \le \frac{1}{4c}\theta_m^2\}$ which specifies the bound of $V_0(t)$ in the finite interval $[0,\ T]$. The boundedness of $V_0(t)$ implies the boundedness of e_0, in the sequel the boundedness of x_0, ξ_0, and $\hat{\theta}_0 = -\xi_0 e_0$. ■

Appendix B: Proof of Theorem 1

Proof: Note that conditions a)-c) are special cases of the condition d), thus we need only to consider the condition d). We will prove this property by the Mathematical Induction method.

Define the following Lyapunov functional

$$V(e_i, \phi_i, \phi_{i-1}, t) = \frac{1}{2}e_i^2(t) + \frac{1}{2}\int_0^t \phi_i^2(\tau)d\tau + \frac{1}{2}\int_t^T \phi_{i-1}^2(\tau)d\tau. \quad (12)$$

The upper right hand derivative of $V(e_i, \phi_i, \phi_{i-1}, t)$ is

$$\dot{V}(e_i, \phi_i, \phi_{i-1}, t) = e_i\dot{e}_i + \frac{1}{2}(\phi_i^2 - \phi_{i-1}^2). \quad (13)$$

Substituting the closed-loop error dynamics (6), the first term on the right hand side of (13) is

$$e_i\dot{e}_i = -\phi_i\xi_i e_i - ke_i^2. \quad (14)$$

Next substituting the parametric learning law (5) into the second term on the right hand side of (13), using the relations $(a-b)^2 - (a-c)^2 = -2(a-b)(b-c) - (b-c)^2$ and the property $(\theta - \hat{\theta})^2 \ge (\theta - \text{proj}(\hat{\theta}))^2$ for any $\hat{\theta}$, we have

$$\begin{aligned} \frac{1}{2}(\phi_i^2 - \phi_{i-1}^2) &= \frac{1}{2}[(\theta - \hat{\theta}_i)^2 - (\theta - \hat{\theta}_{i-1})^2] \\ &\le \frac{1}{2}[(\theta - \hat{\theta}_i)^2 - (\theta - \text{proj}(\hat{\theta}_{i-1}))^2] \\ &= -(\theta - \hat{\theta}_i)(\hat{\theta}_i - \text{proj}(\hat{\theta}_{i-1})) \\ &\quad -\frac{1}{2}(\hat{\theta}_i - \text{proj}(\hat{\theta}_{i-1}))^2 \\ &= \phi_i\xi_i e_i - \frac{1}{2}\xi_i^2 e_i^2. \end{aligned} \quad (15)$$

Clearly $\phi_i\xi e_i$ appears in (14) and (15) with opposite signs. Therefore, the upper right hand derivative of $V(e_i, \phi_i, \phi_{i-1}, t)$ is

$$\dot{V}(e_i, \phi_i, \phi_{i-1}, t) = -ke_i^2 - \frac{1}{2}\xi_i^2 e_i^2 < 0. \quad (16)$$

Integrating the derivative of V, using the negativeness of $\dot{V}$, the boundedness of e_i and $\hat{\theta}_i$ can be derived if $V(e_i(0), \phi_i(0), \phi_{i-1}(0))$ is bounded, i.e.

$$\begin{aligned} & V(e_i(t), \phi_i(t), \phi_{i-1}(t), t) \\ = \ & V(e_i(0), \phi_i(0), \phi_{i-1}(0), 0) + \int_0^t \dot{V}dt \\ \le \ & V(e_i(0), \phi_i(0), \phi_{i-1}(0), 0). \end{aligned} \quad (17)$$

Note that

$$V(e_i(0), \phi_i(0), \phi_{i-1}(0), 0) = \frac{1}{2}e_i^2(0) + \frac{1}{2}\int_0^T \phi_{i-1}^2(\tau)d\tau$$

and $e_i(0)$ is always bounded by the initial condition d).

Let us look at the first iteration $i = 1$,

$$V(e_1(0), \phi_1(0), \phi_0(0), 0) = \frac{1}{2}e_1^2(0) + \frac{1}{2}\int_0^T \phi_0^2(\tau)d\tau$$

is bounded because $\phi_0(t)$ is bounded according to Proposition 1. In the sequel $V(e_1(t), \phi_1(t), \phi_0(t), t) \le V(e_1(0), \phi_1(0), \phi_0(0), 0)$ is bounded. From the parametric learning law (5), the boundedness of e_1 warrants the boundedness of $\hat{\theta}_1$.

Now assume that $(e_{i-1}, \hat{\theta}_{i-1})$ are bounded for all $t \in [0,\ T]$, so is $V(e_i(0), \phi_i(0), \phi_{i-1}(0), 0)$. From (17),

$V(e_i(t), \phi_i(t), \phi_{i-1}(t), t)$ is bounded. Similarly, from the boundedness of e_i and the parametric learning law (5) we can derive the boundedness of $\hat{\theta}_i$. By the Mathematical Induction, the quantities $(e_i, \hat{\theta}_i)$ are bounded for any $i \geq 0$. ■

Appendix C: Proof of Proposition 2

Proof: The difference between V_i and V_{i-1} is

$$\Delta V_i = V_i - V_{i-1} = \frac{1}{2}e_i^2 + \int_0^t (\phi_i^2 - \phi_{i-1}^2)d\tau - \frac{1}{2}e_{i-1}^2. \quad (18)$$

Substituting the control law (4) and the error dynamics (6), the first term on the right hand side of (18) is

$$\begin{aligned} \frac{1}{2}e_i^2 &= \int_0^t e_i \dot{e}_i d\tau + \frac{1}{2}e_i^2(0) \\ &= \int_0^t (-\phi_i \xi_i e_i - k e_i^2)d\tau + \frac{1}{2}e_i^2(0). \end{aligned}$$

Similarly as (15), the second term on the right hand side of (18) can be expressed as

$$\frac{1}{2}\int_0^t (\phi_i^2 - \phi_{i-1}^2)d\tau \leq \int_0^t (\phi_i \xi_i e_i - \frac{1}{2}\xi_i^2 e_i^2)d\tau.$$

Therefore, the difference becomes

$$\begin{aligned} \Delta V_i &\leq -\int_0^t k e_i^2 d\tau - \frac{1}{2}\int_0^t \xi_i^2 e_i^2 d\tau \\ &\quad -\frac{1}{2}e_{i-1}^2(t) + \frac{1}{2}e_i^2(0). \end{aligned} \quad (19)$$

Applying (19) repeatedly we have

$$\begin{aligned} V_i(t) &= V_0(t) + \sum_{j=1}^{i} \Delta V_j \\ &\leq V_0(t) + \frac{1}{2}\sum_{j=1}^{i} e_j^2(0) \\ &\quad - \sum_{j=1}^{i}\int_0^t k e_j^2 d\tau - \frac{1}{2}\sum_{j=1}^{i-1} e_j^2(t), \end{aligned}$$

consequently

$$\begin{aligned} \lim_{i\to\infty} V_i(t) &\leq V_0(t) + \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i} e_j^2(0) \\ &\quad - \lim_{i\to\infty}\sum_{j=1}^{i}\int_0^t k e_j^2 d\tau - \lim_{i\to\infty}\frac{1}{2}\sum_{j=1}^{i-1} e_j^2(t). \end{aligned}$$

■

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