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Repetitive Learning Control: Existence of Solution, Convergence and Robustification

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***Abstract*— In this paper, we propose a repetitive learning control (RLC), which deals with nonlinear dynamical systems with non-parametric uncertainties. We address three fundamental issues associated with the new learning control methods: the existence of the solution, learning convergence property and robustification, which are indispensable for the learning control methods to evolve to a new paradigm. Applying the existence theorem of the differential difference equation of neutral type, and using Lyapunov-Krasovskii functional, the existence of solution and the learning convergence can be proven rigorously. To enhance the robustness of the repetitive learning control, we further develop two kinds of robustification methods with projection and damping respectively to ensure the boundedness of the learning signals.**

I. Introduction

Learning control aims at achieving the desired system performance via directly updating the control input, either repeatedly over a fixed finite time interval, or repetitively (cyclically) over an infinite time interval.

The concept of repetitive control was first proposed in [1] for LTI systems and the convergence analysis was conducted in frequency domain using small gain theorem. In [2] and [3], the stability analysis was conducted in the form of differential-difference equations for linear repetitive processes. In [4], some design issues were exploited for linear repetitive control. In [5], an adaptive feedforward control using internal model equivalence was developed, which deals with LTI systems with an exogenous disturbance consisting of a finite number of sinusoidal functions, and the adaptation mechanism estimates the constant unknown coefficients.

The extension of repetitive control to nonlinear dynamics has also been exploited. In [6], the learning control has been applied to identify and compensate for a nonlinear disturbance function which is represented as an integral of a predefined kernel function multiplied by an unknown influence function that is state independent. In [7], a kind of adaptive learning control scheme was proposed for a class of feedback linearizable systems to track a periodic reference, and the problem can be converted into the learning of a finite number of Fourier coefficients. In [8], the repetitive learning control is applied to a class of nonlinear systems with matched periodic disturbance. Since the periodic disturbance is a time function, it can also be treated as an unknown periodic coefficient under the framework of adaptive control [9]. Note that, above mentioned learning control schemes require the plant to be parameterizable and what is aimed is asymptotic convergence along the time horizon, hence they may also be regarded as some kinds of nonlinear adaptive control under the generalized framework of adaptive control theory. In [10], a repetitive learning control scheme was developed for nonlinear dynamics without parameterization. Nonlinear robust control is used together with the repetitive learning mechanism, hence it requires the upper bound knowledge of the lumped uncertaities.

Under the present theoretical framework of repetitive control, it would be difficult to deal with plants with unknown nonlinear components that are not parameterizable. Henceforth, our objective in this work is to establish a new control strategy – repetitive learning control (RLC) for nonlinear systems with non-parametric uncertainties. The learnability of the traditional repetitive control, acquired via the delay-loop, can be retained by incorporating such a delay-loop into a nonlinear learning mechanism. Meanwhile, a nonlinear feedback law will have to be developed to stabilize the nonlinear dynamics.

This paper is organized as follows. In Section II, the repetitive learning control problem is formulated first. Then the existence of solution and learning convergence properties are analyzed in Section III. Section IV presents two robustified RLC schemes. The conclusion is given in Section V.

Notations defined below will be used throughout the paper. $\|\cdot\|$ is a vector norm. $|y(t)|_s = \max_t |y(t)|$ for any scalar y. $\|\mathbf{y}(t)\|_s = \max_t \|\mathbf{y}(t)\|$ for any vector $\mathbf{y}$. λ_A is the minimum eigenvalue of the matrix A. $\mathcal{C}([a,b];\mathcal{R}^m)$ is the space of continuous functions from $[a,b]$ to $\mathcal{R}^m$. $\mathcal{C}^1([a,b];\mathcal{R}^m)$ is the space of continuously differentiable functions from $[a,b]$ to $\mathcal{R}^m$. $\mathcal{C}^n_{PT}([a,b];\mathcal{R}^m)$ denotes the space of $n-$ order continuously differentiable and periodic functions with periodicity T: $\mathbf{f}(t) = \mathbf{f}(t-T)$ and the mapping $\mathbf{f} : [a,b]$ to $\mathcal{R}^m$. Let $\mathbf{f}_\perp = \mathbf{f}(t-T)$.

II. Problem Formulation

Consider the following system

$$\begin{aligned} \dot{x}_j &= x_{j+1}, & j = 1,2,\cdots,n-1, \\ \dot{x}_n &= \eta(t,\mathbf{x}) + u(t), & \mathbf{x}(0) = \mathbf{x}_0, \end{aligned} \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$, and $\eta(t,\mathbf{x})$ is a continuously differentiable function w.r.t. the arguments $\mathbf{x}$ and t. In particular $\eta(t,\mathbf{x})$ is a lumped, non-parameterizable, and

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Local Lipschitzian nonlinear function, for example, $\eta(t, \mathbf{x}) = x^2 \cos x$ or $\eta(t, \mathbf{x}) = \frac{x_2}{2+\sin t + x_1^2}$.

The control objective is to track the target trajectory $\mathbf{x}_r(t)$ generated by

$$\begin{aligned} \dot{x}_{r,j} &= x_{r,j+1}, \qquad j = 1, 2, \cdots, n-1, \\ \dot{x}_{r,n} &= s(t, \mathbf{x}_r, r), \qquad \mathbf{x}_r(0) \end{aligned} \tag{2}$$

where $\mathbf{x}_r = [x_{r,1}, x_{r,2}, \cdots, x_{r,n}]^T$, $s(\mathbf{x}_r, r, t)$ is a known smooth function w.r.t. all arguments, r is a constant reference input, and $\mathbf{x}_r(0)$ is a vector of the initial states. The ideal control input, $u_r(t)$, can be computed directly from the relation

$$\dot{x}_{r,n}(t) = \eta(t, \mathbf{x}_r) + u_r(t) \tag{3}$$

with the initial values $\mathbf{x}_r(0)$. From (2), $\dot{x}_{r,n} = s(t, \mathbf{x}_r(t), r)$. Therefore the ideal control is $u_r(t) = s(\mathbf{x}_r(t), t, r) - \eta(t, \mathbf{x}_r)$, which is however not available because of the presence of the unknown $\eta(t, \mathbf{x}_r)$. The central task now is to learn the ideal control $u_r(t)$. As such, the learning objective shall be the quantity $u_r(t)$, that is, to learn the ideal control profile directly. As being known, the repetitive learning control is especially effective in dealing with periodic quantities. Thus if $u_r(t)$ is periodic, we may apply the repetitive learning control approach to solve the problem.

Assumption 1: The desired trajectory $\mathbf{x}_r(t)$, and the quantity $\eta(t, \mathbf{x}_r)$, are periodic with a periodicity T, namely, $\mathbf{x}_r(t) \in \mathcal{C}^2_{PT}([0, \infty); \mathcal{R}^n)$ and $\eta(t, \mathbf{x}_r) = \eta(t - T, \mathbf{x}_r)$.

Remark 1: Any homogeneous function $\eta(\mathbf{x})$ satisfies Assumption 1.

From the periodicity of $\mathbf{x}_r(t)$, we can derive that $\dot{\mathbf{x}}_r \in \mathcal{C}^1_{PT}([0, \infty); \mathcal{R}^n)$ and $s(t, \mathbf{x}_r(t), r) \in \mathcal{C}^1_{PT}([0, \infty); \mathcal{R}^1)$. From the periodicity of $\mathbf{x}_r(t)$ and Assumption 1, $\eta(t, \mathbf{x}_r) \in \mathcal{C}^1_{PT}([0, \infty); \mathcal{R}^1)$. In the sequel, the ideal control $u_r(t) = s(t, \mathbf{x}_r(t), r) - \eta(t, \mathbf{x}_r)$, is a function in the space $\mathcal{C}^1_{PT}([0, \infty); \mathcal{R}^1)$. The principal idea of repetitive learning control method, therefore, shall be applicable for this class of periodic learning tasks.

We may note the discrepancy in initial conditions $\mathbf{x}(0) \neq \mathbf{x}_r(0)$. Even if $u_r(t)$ is directly achievable such that $u(t) = u_r(t)$ for $t \geq 0$, the nonlinear system (1) may not produce the desired response $\mathbf{x}_r$, what is more, it may even go divergence in a finite time. From the theory of differential equation, a nonlinear ODE may produce totally different solution trajectories under different initial conditions. We need a robust control mechanism working concurrently with the learning mechanism to guarantee the asymptotic stability of the closed-loop system. In designing a robust feedback controller for the nonlinear system (1), the most popular approach is first to assume a upper bounding function $\alpha(t, \mathbf{x})$ for $\eta(t, \mathbf{x})$, e.g. $|\eta(t, \mathbf{x})| \leq \alpha(t, \mathbf{x})$. The min-max control [11] and sliding mode control [12] are representative approaches of robust feedback control. Repetitive learning can be incorporated into the robust control loop [10]. However, it should be noted that the robust control alone can work well in this circumstance, and the learning mechanism is an add-on to the existing robust control aiming at further improving the performance. In this work, we explore a new scenario in which the robust control alone is unable to ensure a stable closed-loop, thus the repetitive learning mechanism and the robust control mechanism have to be integrated, working jointly to warrant a stable control loop and meanwhile achieve learning convergence repetitively.

The new scenario is characterized by the following bounding condition.

Assumption 2:

$$|\eta(t, \mathbf{x}) - \eta(t, \mathbf{x}_r)| \leq \alpha(t, \mathbf{x}, \mathbf{x}_r)\|\mathbf{x} - \mathbf{x}_r\|,$$

where $\alpha(t, \mathbf{x}, \mathbf{x}_r)$ is a known bounding function.

Assumption 2 implies that the "variation" of the local Lipschitzian function η with respect to $\mathbf{x}$ should be limited from above by a known bound which can also be any nonlinear function, e.g. local Lipschitzian function, of $\mathbf{x}$. Hence it is not a very strict constraint. Clearly, most existing robust control methods may not be suitable in this circumstance because a bound for the variation of η does not warrant a finite bound for η itself.

Let us construct the integrated controller. First formulate the error dynamics of $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_r$. Define $\mathbf{b} = [0\ 0\ \cdots\ 0\ 1]^T$, and $\mathbf{c} = [c_1, c_2, \cdots, c_{n-1}, 1]$ is chosen such that

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -1 \end{bmatrix} \tag{4}$$

is an asymptotically stable matrix. Based on Lyapunov stability theory for LTI systems, for a given positive definite matrix $Q \in R^{n\times n}$, there exits a unique positive definite matrix $P \in R^{n\times n}$ satisfying the following Lyapunov equation $A^T P + PA = -Q$. Let λ_Q be the minimum eigenvalue of the matrix Q, $-\mathbf{w}^T Q \mathbf{w} \leq -\lambda_Q \|\mathbf{w}\|^2$ holds for any $\mathbf{w} \in R^n$.

From (1) and (3), the dynamics of $\Delta\mathbf{x}$ can be expressed as

$$\Delta\dot{\mathbf{x}} = A\Delta\mathbf{x} + \mathbf{b}(\mathbf{c}\Delta\mathbf{x} + \eta - \eta_r + u - u_r), \tag{5}$$

where $\eta_r = \eta(t, \mathbf{x}_r)$. The integrated repetitive learning control law is

$$u(t) = \hat{u}(t) - \mathbf{c}\Delta\mathbf{x} - \frac{1}{\lambda_Q}\alpha^2(t, \mathbf{x}, \mathbf{x}_r)\sigma(t), \tag{6}$$

$$\hat{u}(t) = \hat{u}(t - T) - k(t)\sigma(t), \tag{7}$$

$$\hat{u}(t) = 0, \forall t \in [-T,\ 0],$$

where $\sigma(t) = \mathbf{b}^T P \Delta\mathbf{x}$. $k(t)$ is the learning gain defined as

$$k(t) = \begin{cases} 0, & -T \leq t < 0, \\ k_1(t), & 0 \leq t < T, \\ q, & t \geq T, \end{cases} \tag{8}$$

where $q > 0$ is a constant, $k_1(t)$ is chosen to be monotone and smooth such that $k(t)$ is a smooth function on $[-T, \infty)$.

Note that now the objective of repetitive learning is to directly learn the ideal control, that is, tune $\hat{u}(t)$ in (7) to approach $u_r(t)$. $-\frac{1}{\lambda_Q}\alpha^2(t,\mathbf{x},\mathbf{x}_r)\sigma(t)$ in (6) constitutes the robust feedback.

Proposition 1: [13] Consider the following Cauchy problem

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}), \quad \mathbf{x}(t_0)=\mathbf{x}_0. \tag{9}$$

Suppose that $\mathbf{f}(t,\mathbf{x})$ is continuous for $(t,\mathbf{x})$ in a region Ω, and satisfies the local Lipschitz condition with respect to $\mathbf{x}$. Then the solution of Cauchy problem (9) can be continued to the boundary, $\partial\Omega$, of Ω (possible ∞).

According to [14] and [15] (Chapter 5, §12), we have the following proposition:

Proposition 2: Consider the following differential difference equation of neutral type

$$\dot{\mathbf{x}}(t)=\mathbf{f}(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\dot{\mathbf{x}}(t-\tau)), \qquad t\geq t_0,$$

where the retardation τ is assumed constant. If the function $\mathbf{f}$ is continuous for the arguments, and the initial function $\mathbf{x}_0$ has a continuous derivation for $t_0-\tau\leq t\leq t_0$, then the solution $\mathbf{x}$ exists in the neighborhood of the point $t=t_0$.

III. Existence of Solution and Convergence

Substituting the learning control law (6) into the dynamics (5), the closed-loop error dynamics is

$$\Delta\dot{\mathbf{x}} = A\Delta\mathbf{x}+\mathbf{b}(\eta-\eta_r-\nu-\frac{1}{\lambda_Q}\alpha^2\sigma). \tag{10}$$

where $\alpha=\alpha(t,\mathbf{x},\mathbf{x}_r)$ and $\nu=u_r-\hat{u}$. In the closed-loop dynamics, there are two unknown terms u_r and $\eta-\eta_r$. The first term will be compensated by $\hat{u}$ through repetitive learning. The second term $\eta-\eta_r$ will be compensated jointly by $A\Delta\mathbf{x}$ and the robust control $-\frac{1}{\lambda_Q}\alpha^2\sigma$.

From the error dynamics (10) and the updating law (6), we have

$$\begin{aligned}\Delta\dot{\mathbf{x}} &= \mathbf{f}(t,\Delta\mathbf{x},\hat{u})\\ \hat{u}(t) &= \hat{u}(t-T)-k(t)\mathbf{b}^TP\Delta\mathbf{x},\end{aligned} \tag{11}$$

where $\mathbf{f}(t,\Delta\mathbf{x},\hat{u})=A\Delta\mathbf{x}+\mathbf{b}(\eta-\eta_r+\hat{u}-u_r-\frac{1}{\lambda_Q}\alpha^2\sigma)$. The learning control system consists of differential and continuous-time difference equations of neutral type.

Theorem 1: For the system (11) under Assumption 1 and Assumption 2, the learning control law (6) and (7) guarantees the existence of solution $(\Delta\mathbf{x},\hat{u})$ in $[0,\infty)$ and asymptotical convergence $\lim_{t\to\infty}\int_{t-T}^{t}\|\Delta\mathbf{x}(\tau)\|^2d\tau$.

Proof: Define the regions $\Omega_i \triangleq [(i-1)T,\ iT)\times R^n$ for $(t,\mathbf{x})$. The proof is composed of three parts. *Part 1* and *Part 2* prove the existence of solution $(\Delta\mathbf{x},\hat{u})$ in the domain $[0,\ T)$ and $[T,\ \infty)$ respectively. *Part 3* derives the convergence property of the tracking error $\Delta\mathbf{x}$.

Part 1. Existence of the solution $(\Delta\mathbf{x},\hat{u})$ *in* $[0,\ T)$

For $i=1$, we have $\hat{u}(t)=\mathbf{0}$ for $t\in[-T,\ 0]$. Therefore, by substituting $\hat{u}(t)$ into $\mathbf{f}$ the dynamics (11) renders to a set of ODE (Ordinary Differential Equation), and $\mathbf{f}(t,\Delta\mathbf{x},\hat{u}):\Omega_1\to R^n$ is continuous in $\Delta\mathbf{x}$ by virtue of the smoothness of η. By Peano's Existence Theorem [13], associated with the initial condition $\Delta\mathbf{x}(0)$, the equation (11) has a continuous solution in a neighborhood of $t=0$. Furthermore it is easy to check that $\mathbf{f}(t,\Delta\mathbf{x},\hat{u})$ is locally Lipschitzian in $\Delta\mathbf{x}$. We need only to consider the solution for $t>0$. Assume $[0,t_1)$ be the maximal interval to which the solution $\Delta\mathbf{x}$ can be continued up. Proposition 1 implies that $\Delta\mathbf{x}$ tends to the boundary $\partial\Omega_1$ of Ω_1 as $t\to t_1$. It further implies that $\lim_{t\to t_1}\|\Delta\mathbf{x}\|=\infty$ if $t_1\leq T$, i.e., for any $C>0$, there exists $\delta_1>0$ such that $\|\Delta\mathbf{x}\|\geq C$ for all $t\geq t_1-\delta_1$. Since $\Delta\mathbf{x}$ exists for all $t\in[0,t_1-\frac{\delta_1}{2}]$, define the following Lyapunov-Krasovskii functional:

$$V(t,\Delta\mathbf{x},\nu)=\frac{1}{2}\Delta\mathbf{x}^TP\Delta\mathbf{x}+\frac{1}{2q}\int_{t-T}^{t}\nu^2(\tau)d\tau. \tag{12}$$

Now we prove the finiteness of $V(t,\Delta\mathbf{x},\nu)$ for all $t\in[0,t_1-\frac{\delta_1}{2}]$. From the existence theorem of differential equation [16] there exists a $T_1>0$ and $[0,T_1)\subset[0,t_1-\frac{\delta_1}{2}]$, the boundedness of $V(t,\Delta\mathbf{x},\nu)$ over $[0,T_1)$ can be guaranteed and we need only focus on the interval $[T_1,t_1-\frac{\delta_1}{2}]$. For any $t\in[T_1,t_1-\frac{\delta_1}{2}]$, the upper right hand derivative of V is $\dot{V}=\frac{1}{2}(\Delta\dot{\mathbf{x}}^TP\Delta\mathbf{x}+\Delta\mathbf{x}^TP\Delta\dot{\mathbf{x}})+\frac{1}{2q}(\nu^2-\nu_\perp^2)$, where $\nu_\perp=u_{r,\perp}-\hat{u}_\perp$, $u_{r,\perp}=u_r(t-T)$ and $\hat{u}_\perp=\hat{u}(t-T)$. Substitution of the tracking error dynamics (10) yields

$$\begin{aligned}&\frac{1}{2}(\Delta\dot{\mathbf{x}}^TP\Delta\mathbf{x}+\Delta\mathbf{x}^TP\Delta\dot{\mathbf{x}})\\ &=-\frac{1}{2}\Delta\mathbf{x}^TQ\Delta\mathbf{x}+\sigma(\eta-\eta_r-\nu-\frac{1}{\lambda_Q}\alpha^2\sigma)\\ &\leq-\frac{\lambda_Q}{2}\|\Delta\mathbf{x}\|^2+|\sigma|\cdot\alpha\|\Delta\mathbf{x}\|-\frac{1}{\lambda_Q}\alpha^2\sigma^2-\sigma\nu\\ &=-\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2-\sigma\nu\\ &-(\frac{\sqrt{\lambda_Q}}{2}\|\Delta\mathbf{x}\|-\frac{1}{\sqrt{\lambda_Q}}\alpha|\sigma|)^2.\end{aligned} \tag{13}$$

Since $\hat{u}_\perp=\hat{u}(t-T)=0$ for all $t\in[0,\ T)$, we have $\hat{u}(t)=-k_1(t)\sigma(t)$. From the definition of $k(t)$, $k_1(t)$ is strictly increasing in $[0,T)$, thus $\frac{1}{k_1(t)}\geq\frac{1}{q}$ is ensured in the time interval $[T_1,\ T)$. We have

$$\begin{aligned}\frac{1}{2q}(\nu^2-\nu_\perp^2) &= \frac{1}{2q}(u_r-\hat{u})^2-\frac{1}{2q}(u_{r,\perp}-\hat{u}_\perp)^2\\ &\leq \frac{1}{2k_1(t)}(u_r-\hat{u})^2\\ &\leq \frac{u_r^2}{2k_1(t)}-\frac{1}{k_1(t)}\hat{u}(u_r-\hat{u})-\frac{\hat{u}^2}{2k_1(t)}\\ &\leq \frac{u_r^2}{2k_1(t)}+\sigma\nu.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\dot{V} &\leq -\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2-(\frac{\sqrt{\lambda_Q}}{2}\|\Delta\mathbf{x}\|-\frac{1}{\sqrt{\lambda_Q}}\alpha|\sigma|)^2\\ &\quad -\sigma\nu+\frac{u_r^2}{2k_1(t)}+\sigma\nu\leq\frac{u_r^2}{2k_1(t)},\end{aligned} \tag{14}$$

i.e., $V(t,\Delta\mathbf{x},\nu)\leq V(T_1,\Delta\mathbf{x}(T_1),\nu(T_1))+\frac{1}{2}\int_{T_1}^{t}\frac{u_r^2(\tau)}{k_1(\tau)}d\tau$. Since $u_r(t)\in\mathcal{C}^1_{PT}([0,\infty);\mathcal{R}^1)$, $\int_{T_1}^{t}\frac{u_r^2(\tau)}{k_1(\tau)}d\tau$ is bounded. Thus V is bounded for all $t\in[0,t_1-\frac{\delta_1}{2}]$. Let $N^2\lambda_P>0$

be the bound of V on $[0, t_1 - \frac{\delta_1}{2}]$, where λ_P is the minimum eigenvalue of the positive definite matrix P. Then N does not depend on δ_1. By the definition of Lyapunov functional V, we can see that $\|\Delta\mathbf{x}\| \leq \sqrt{V/\lambda_P} = N$ for all $t \in [0, t_1 - \frac{\delta_1}{2}]$. Taking $C = 2N$ in advance, for the corresponding $\delta_1 > 0$ we have $C \leq \|\Delta\mathbf{x}(t_1 - \frac{\delta_1}{2})\| \leq N = \frac{C}{2}$, a contradiction which implies $t_1 \geq T$. This assures the solution $\Delta\mathbf{x}$ of the dynamic system (11) exists in $[0, T]$. Further, considering the smoothness of the right hand side of equation (11), $\Delta\mathbf{x}(t)$ and $\hat{u}(t)$ are both continuously differentiable for any $t \in [0, T)$.

Part 2. Existence of the solution $(\Delta\mathbf{x}, \hat{u})$ *in* $[T, \infty)$

Assume that the solution $\Delta\mathbf{x}$ and $\hat{u}$ of the differential difference equation (11) exists in $[(j-1)T, jT)$ for $j = 2, \cdots, i-1$. This implies both $\mathbf{x}$ and $\hat{u}$ are continuously differentiable for all $t \in [0,\ (i-1)T)$. Assume that the solution of (11) can be continued up to a time $t \in [(i-1)T, iT)$, by differentiating $\hat{u}$ we obtain

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \mathbf{f}(t, \Delta\mathbf{x}, \hat{u}), \qquad t \in [(i-1)T, iT), \\ \dot{\hat{u}}(t) &= g(t, \Delta\mathbf{x}, \hat{u}(t), \dot{\hat{u}}(t-T)), \end{aligned} \tag{15}$$

where

$$g(t, \Delta\mathbf{x}, \hat{u}(t), \dot{\hat{u}}(t-T)) = \dot{\hat{u}}(t-T) - q\mathbf{b}^T P\mathbf{f}(t, \Delta\mathbf{x}, \hat{u}).$$

The function $\mathbf{f}(t, \Delta\mathbf{x}, \hat{u})$ and $g(t, \Delta\mathbf{x}, \hat{u}(t), \dot{\hat{u}}(t-T))$ are continuous with respect to the arguments $\Delta\mathbf{x}$ and $\hat{u}$ on $[(i-2)T, (i-1)T)$. For $t > T$, $\hat{u}_\perp$ cannot be ignored in the updating law, and (15) is now truly a mixture of differential and continuous-time difference equations of neural type. According to Proposition 2, the solution $(\Delta\mathbf{x},\ \hat{u})$ of the equation (15) exists at the neighborhood of the point $(i-1)T$. Furthermore, $\mathbf{f}(t, \Delta\mathbf{x}, \hat{u}) : \Omega_i \rightarrow R^n$ is continuous and locally Lipschitzian in $\Delta\mathbf{x}$ and $\hat{u}$. Thus the solution $\Delta\mathbf{x}$ can be continued up to the boundary $\partial\Omega_i$ of Ω_i. Let $[(i-1)T, t_i)$ be the maximal interval to which the solution $\Delta\mathbf{x}$ can be continued up. If $t_i \leq iT$, there exists a $\delta_i > 0$ such that $\|\Delta\mathbf{x}\| \geq C$ for all $t \geq t_i - \delta_i$. For $t \in [(i-1)T, t_i - \frac{\delta_i}{2})$, define the Lyapunov-Krasovskii functional

$$V(t, \Delta\mathbf{x}, \nu) = \frac{1}{2}\Delta\mathbf{x}^T P\Delta\mathbf{x} + \frac{1}{2q}\int_{t-T}^{t} \nu^2 d\tau. \tag{16}$$

Then the upper right hand derivative of V is

$$\dot{V} = \frac{1}{2}(\Delta\dot{\mathbf{x}}^T P\Delta\mathbf{x} + \Delta\mathbf{x}^T P\Delta\dot{\mathbf{x}}) + \frac{1}{2q}(\nu^2 - \nu_\perp^2) \tag{17}$$

For the first term on the right side of (17), the result of (13) still holds. Let us compute the second term on the right hand side of (17). Using the learning updating law (7), the periodic property $u_r = u_{r,\perp}$, and the algebraic relationship

$$(a-b)^2 - (a-c)^2 = -2(a-b)(b-c) - (b-c)^2 \tag{18}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors with the same dimensions, we have

$$\begin{aligned} &\frac{1}{2q}[(u_r - \hat{u})^2 - (u_{r,\perp} - \hat{u}_\perp)^2] \\ &= \frac{1}{2q}[-2(u_r - \hat{u})(\hat{u} - \hat{u}_\perp) - (\hat{u} - \hat{u}_\perp)^2] \\ &= \sigma\nu - \frac{q}{2}\sigma^2. \end{aligned} \tag{19}$$

Substituting (13) and (19) into (17), the upper right hand derivative of V is

$$\begin{aligned} \dot{V} \ \leq\ & -\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2 - \frac{q}{2}\sigma^2 \\ & -(\frac{\sqrt{\lambda_Q}}{2}\|\Delta\mathbf{x}\| - \frac{1}{\sqrt{\lambda_Q}}\alpha|\sigma|)^2 \end{aligned} \tag{20}$$

Clearly $V(t, \Delta\mathbf{x}, \nu)$ will be bounded for $t \in [(i-1)T,\ t_i - \frac{1}{2}\delta_i)$ as far as $V(\tau, \Delta\mathbf{x}(\tau), \nu(\tau))$ is bounded for $\tau \in [0,\ (i-1)T)$. Let $N^2\lambda_P$ be the bound of V on $[(i-1)T, t_i - \frac{\delta_i}{2})$, then N does not depend on δ_i. By the definition of Lyapunov-Krasovskii functional, we have $\|\Delta\mathbf{x}(t)\| \leq \sqrt{V/\lambda_P} = N$ for all $t \in [(i-1)T,\ t_i)$. Taking $C = 2N$ in advance, if the solution can only be continued up to $t_i < iT$, then we again has the contradiction $C \leq \|\Delta\mathbf{x}(t_i - \frac{\delta_i}{2})\| \leq N = \frac{C}{2}$. According to the theory of mathematical induction, the solution $\Delta\mathbf{x}$ exists in $t \in [(i-1)T,\ iT)$ for any finite i. Furthermore, since the solution $\hat{u}(t)$ exists for $t \in [0,\ (i-1)T)$, then from $\hat{u}(t) = \hat{u}(t-T) + k(t)\mathbf{b}^T P\Delta\mathbf{x}$ and the existence of $\Delta\mathbf{x}$ for $t \in [(i-1)T, iT)$, the solution $\hat{u}$ exists for $t \in [(i-1)T,\ iT)$. Thus the solution $(\Delta\mathbf{x}, \hat{u})$ exists in $[0,\ iT)$ for any finite i. This implies that the solution $(\Delta\mathbf{x}, \hat{u})$ either is uniformly bounded or tends to infinity as $t \rightarrow \infty$. Thus $\Delta\mathbf{x}$ and $\hat{u}$ exist for $t \in [0, \infty)$.

Part 3. Asymptotical convergence

Now derive the integral convergence

$$\lim_{t \to \infty} \int_{t-T}^{t} \|\Delta\mathbf{x}\|^2 d\tau = 0$$

using the relation (20), that is, $\dot{V}$ is negative semi-definite for $t \in [T, \infty)$. Suppose that $\lim_{t\to\infty} \int_{t-T}^{t} \|\Delta\mathbf{x}\|^2 d\tau \neq 0$. Then there exist an $\varepsilon > 0$, $t_m \geq T$ and a sequence $t_i \rightarrow \infty$ with $i = 1, 2, \cdots$ and $t_{i+1} \geq t_i + T$ such that $\int_{t_i - T}^{t_i} \|\Delta\mathbf{x}\|^2 d\tau > \varepsilon$ when $t_i > t_m$. Hence from (20), we obtain

$$\begin{aligned} \lim_{t\to\infty} V(t, \Delta\mathbf{x}, \nu) \ \leq\ & V(T, \Delta\mathbf{x}(T), \nu(T)) \\ & - \lim_{i\to\infty} \sum_{j=1}^{i} \int_{t_j - T}^{t_j} \|\Delta\mathbf{x}(\tau)\|^2 d\tau \end{aligned}$$

Since $V(T, \Delta\mathbf{x}(T), \nu(T))$ is finite, the above relation implies $\lim_{t\to\infty} V(t, \Delta\mathbf{x}, \nu) = -\infty$, a contradiction to the non-negativeness property of Lyapunov-Krasovskii functional $V(t, \Delta\mathbf{x}, \nu) \geq 0$. ■

IV. ROBUSTIFICATION

A. Learning Control With Projection

From the point of view of practical implementation, $u_r(t)$ must be finite. If there exists a known constant u^* such that for the given $\mathbf{x}_r(t)$, $\max_t |u_r(t)| \le u^*$, the updating law (7) can be modified as

$$\begin{aligned}\hat{u}(t) &= \mathcal{P}(\hat{u}(t-T)) - k(t)\sigma(t),\\ \hat{u}(t) &= 0, \quad \forall t \in [-T,\ 0],\end{aligned} \tag{21}$$

where the projection operator $\mathcal{P}(\hat{u})$ is defined as

$$\mathcal{P}(\hat{u}) = \begin{cases} \hat{u}, & |\hat{u}| \le u^* \\ p(\hat{u}), & |\hat{u}| > u^*, \end{cases}$$

where $p(\hat{\theta}_i) \in \mathcal{C}^1(\mathcal{R};\mathcal{R}^1)$ is a polynomial and satisfying $p(\theta_i^*) = \theta_i^*$, $p(-\hat{\theta}_i) = -p(\hat{\theta}_i)$, $0 \le \frac{\partial p}{\partial \hat{\theta}_i} \le 1$, $\frac{\partial p}{\partial \hat{\theta}_i}|_{\theta_i} = 1$ and the limit $\lim_{\hat{\theta}_i \to \infty} p(\hat{\theta}_i)$ is a constant. Fig.1 shows the shape of the projection operator.

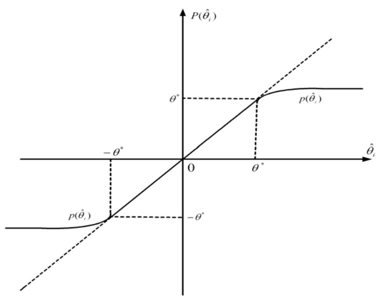

Fig. 1. The definition of $\mathcal{P}(\hat{\theta})$.

With the additional system bounding information, the repetitive learning control achieves improved convergence property, as summarized in the following theorem.

Theorem 2: For the system (1), under Assumption 1 and Assumption 2, the learning control law (6) and (21) guarantees the uniformly asymptotical convergence of $\Delta\mathbf{x}$.

Proof: The solution $(\Delta\mathbf{x}, \hat{u})$ of the dynamic system (11) for $t \in [0, T)$ is the same as Theorem 1 *Part 1* without projection, because $\hat{u}(t-T) = 0$. To prove the existence of solution in $[T, \infty)$, define the same Lyapunov-Krasovskii functional in (16). The relations (17) and (13) still hold as the projection operation is not directly involved. Next look at the relation (19), which might be affected by the introduction of the projection operator.

We can easily verify the property $(u-\hat{u})^2 \ge [u - \mathcal{P}(\hat{u})]^2$, for any quantities $\hat{u}$. Using this property, the updating law (21), the periodic property $u_r = u_{r,\perp}$, and the algebraic relation (18), we have

$$\begin{aligned}&\frac{1}{2q}[(u_r-\hat{u})^2 - (u_{r,\perp} - \hat{u}_\perp)^2]\\ &\le \frac{1}{2q}[(u_r-\hat{u})^2 - (u_r - \mathcal{P}(\hat{u}_\perp))^2]\\ &= \frac{1}{2q}[-2(u_r-\hat{u})(\hat{u} - \mathcal{P}(\hat{u}_\perp)) - (\hat{u} - \mathcal{P}(\hat{u}_\perp))^2]\\ &= \sigma\nu - \frac{q}{2}\sigma^2\end{aligned} \tag{22}$$

which turns out to be the same as (19). In the sequel, the conclusion of *Part 2* in Theorem 1, namely the existence of solution $(\Delta\mathbf{x}, \hat{u})$ over the interval $[T,\ \infty)$, still holds. According to *Part 3* of Theorem 1, the integral convergence of $\Delta\mathbf{x}$, i.e., $\lim_{t\to\infty}\int_{t-T}^{t}\|\Delta\mathbf{x}(\tau)\|^2 d\tau = 0$ is obtained.

By virtue of the projection, the boundedness of $\Delta\mathbf{x}$ ensures the finiteness of $\hat{u}$, thereafter u and $\Delta\dot{\mathbf{x}}$. The boundedness of $\Delta\dot{\mathbf{x}}$ implies the uniform continuity of $\Delta\mathbf{x}$, therein the uniform continuity of the tracking error $\Delta\mathbf{x}$. As a result, $\lim_{t\to\infty}\|\Delta\mathbf{x}(t)\| = 0$. ■

B. Learning With Damping

Learning with damping offers an alternative way when we neither know the bound u^* *a priori* nor want to use a large estimate value as u^*. As such, the updating law (7) can be modified as

$$\begin{aligned}\hat{u}(t) &= \gamma\hat{u}(t-T) - k(t)\sigma(t),\\ \hat{u}(t) &= 0, \quad \forall t \in [-T,\ 0],\end{aligned} \tag{23}$$

where $0 < \gamma \le 1$ is the damping coefficient.

Different from projection, damping is introduced without using any extra system information. Hence it is a trade-off made between the robustness and the tracking convergence.

Theorem 3: For system (1), under Assumption 1 and Assumption 2, the learning control law (6) and (23) guarantees the finiteness of the solution trajectory $(\Delta\mathbf{x}, \hat{u})$ in the large.

Proof: The solution $(\Delta\mathbf{x}, \hat{u})$ for $t \in [0, T)$ is the same as Theorem 1 *Part 1* without damping, because $\hat{u}(t-T) = 0$. Thus in the following we discuss the solution in the interval $[T, \infty)$. Analogous to Theorem 1, assume the solution exists in $[T,\ (i-1)T)$ and can be continued up to $t_i \in [(i-1)T,\ iT)$. We need only to show the finiteness of the solution for any $t_i \in [(i-1)T,\ iT)$. Define the same Lyapunov-Krasovskii functional as (16) in Theorem 1. The relations (17) and (13) still hold as only the closed-loop dynamics is directly involved in the derivation. Next look at the relation (19), which is affected by the introduction of damping. Using the updating law (23), the periodic property $u_r = u_{r,\perp}$ and the algebraic relation (18), we have

$$\begin{aligned}&\frac{1}{2q}(\nu^2 - \nu_\perp^2)\\ =\ &\frac{1}{2q}[(u_r-\hat{u})^2 - (u_r - \hat{u}_\perp)^2]\\ =\ &\frac{1}{2q}[-2(u_r-\hat{u})(\hat{u} - \hat{u}_\perp) - (\hat{u} - \hat{u}_\perp)^2]\\ =\ &-\frac{1}{q}(u_r-\hat{u})(\hat{u} - \gamma\hat{u}_\perp) + \frac{1}{q}(1-\gamma)(u_r-\hat{u})\hat{u}_\perp\\ &-\frac{1}{2q}(\hat{u} - \hat{u}_\perp)^2.\end{aligned} \tag{24}$$

The first term on the right hand side of (24), by substituting the updating law (23), is $\sigma\nu$ which will cancel out the same term but with opposite sign in (13). In order to evaluate last two terms on the right hand side of (24), using the relationship (18) yields

$$\begin{aligned}
&\frac{1}{q}(1-\gamma)(u_r-\hat{u})\hat{u}_\perp-\frac{1}{2q}(\hat{u}-\hat{u}_\perp)^2\\
&\le \frac{1}{2q}(1-\gamma)u_r^2+\frac{1}{2q}(1-\gamma)(\hat{u}-\hat{u}_\perp)^2\\
&-\frac{1}{2q}(1-\gamma)\hat{u}^2-\frac{1}{2q}(\hat{u}-\hat{u}_\perp)^2\\
&\le \frac{1-\gamma}{2q}(u_r^2-\hat{u}^2)-\frac{q\gamma}{2}\sigma^2.
\end{aligned}$$

Therefore, the upper right hand derivative of V is

$$\begin{aligned}
\dot{V} &\le -\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2-(\frac{\sqrt{\lambda_Q}}{2}\|\Delta\mathbf{x}\|-\frac{1}{\sqrt{\lambda_Q}}\alpha|\sigma|)^2\\
&\quad -\frac{q\gamma}{2}\sigma^2+\frac{1-\gamma}{2q}(u_r^2-\hat{u}^2)\\
&\le -\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2-\frac{1-\gamma}{2q}\hat{u}^2+\frac{1-\gamma}{2q}u_r^2 \qquad (25)
\end{aligned}$$

Now we can show the finiteness of V in the interval $[(i-1)T,\ t_i)$. If V is finite at $(i-1)T$, then it remains finite at t_i because $\dot{V}$ is uniformly bounded by $\frac{1-\gamma}{2q}\|u_r\|_s^2$. Consequently $\Delta\mathbf{x}$ and σ remain finite. The finiteness of $\hat{u}$ in the interval $[(i-1)T,\ t_i)$ can be derived from the finiteness of $\sigma(t)$ in (23). This implies the solution $(\Delta\mathbf{x},\hat{u})$ either remains uniformly bounded or tend to infinity as $t\to\infty$. Thus the solution $(\Delta\mathbf{x},\hat{u})$ exists for any $t\in[0,\ \infty)$.

We further show that the solution $(\Delta\mathbf{x},\hat{u})$ remains finite when $t\to\infty$. From (25), $\dot{V}\le 0$ as long as the solution $(\Delta\mathbf{x},\hat{u})$ is outside a compact set $\mathcal{M}$ defined below

$$\mathcal{M}=\left\{(\Delta\mathbf{x},\hat{u}):M(\Delta\mathbf{x},\hat{u})\ge\frac{1-\gamma}{2q}\|u_r\|_s^2\right\},$$

where $M(\Delta\mathbf{x},\hat{u})\stackrel{\triangle}{=}\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2+\frac{1-\gamma}{2q}|\hat{u}|^2$. Define an ϵ-neighbourhood of $\mathcal{M}$ with $\epsilon>0$

$$\mathcal{M}_\epsilon=\left\{(\Delta\mathbf{x},\hat{u}):M(\Delta\mathbf{x},\hat{u})\ge\frac{1-\gamma}{2q}\|u_r\|_s^2+\epsilon\right\},$$

then $\dot{V}\le-\epsilon$ for any $(\Delta\mathbf{x},\hat{u})\in\mathcal{M}_\epsilon^c$ where $\mathcal{M}_\epsilon^c$ is the complementary set of $\mathcal{M}_\epsilon$. Since the solution exists in $[0,\ \infty)$, there is no finite escape time for $(\Delta\mathbf{x},\hat{u})$. First assume that $\Delta\mathbf{x}$, thereby V, diverges asymptotically. Consider the fact that $\dot{V}\le\frac{1-\gamma}{2q}\|u_r\|_s^2$, there must exist an infinite time interval $[t_s,\ \infty)$, such that

$$\frac{\lambda_Q}{4}\|\Delta\mathbf{x}\|^2+\frac{1-\gamma}{2q}|\hat{u}|^2\in\mathcal{M}_\epsilon^c \qquad \forall t\in[t_s,\ \infty).$$

Since the solution exists in $[0,\ \infty)$, $V(t_s,\Delta\mathbf{x}(t_s),\nu(t_s))$ is finite. Integrating $\dot{V}$ in (25) from $t\ge t_s$ we have

$$\lim_{t\to\infty}V(t)\le V(t_s,\Delta\mathbf{x}(t_s),\nu(t_s))-\lim_{t\to\infty}\int_{t_s}^{t}\epsilon d\tau\to-\infty,$$

that is however impossible because $V\ge 0$. We can conclude that $\Delta\mathbf{x}$ cannot stay infinitely long in $\mathcal{M}_\epsilon^c$, and will always re-enter $\mathcal{M}_\epsilon$ after a finite interval. Hence $\Delta\mathbf{x}$ remains finite when $t\to\infty$. Note that the finiteness of $\Delta\mathbf{x}$ warrants the finiteness of $\sigma(t)$ over the entire horizon $[0,\ \infty)$. On the other hand, the learning law (23) with the damping γ is an asymptotically stable first order difference equation subject to the input $k(t)\sigma(t)$. Therefore $\hat{u}$ remains finite when $t\to\infty$. ■

V. CONCLUSION

In this paper, we developed a repetitive learning control, which can effectively handle a class of tracking control problems by making use of the repetitive nature of the control problems. In this class of tracking control problems where no parameterization is available, the target trajectory to be tracked will be any smooth periodic orbit of a reference model, and the system repetitive factor to be learnt is the desired periodic control signals. In this circumstance, the repetitive learning mechanism works jointly with a robust control mechanism to guarantee asymptotical tracking convergence.

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