Maximum singular value and power of an interval matrix

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***Abstract*— Of parametric interval computational problems, the maximum singular value of an interval matrix and the power of an interval matrix at a specified order have not been solved. In this paper, we provide solutions for these problems. First, for the calculation of an exact value of maximum singular value, we make use of the eigenvalue problem of a symmetric interval matrix. Second, we use a sensitivity transfer method to prove that the power of an interval matrix can be found from a set of vertex matrices.**

***Index Terms*— Maximum singular value of an interval matrix, power of an interval matrix.**

I. Introduction

Interval computation techniques [1], [2] are popularly used in the robust stability analysis of uncertain systems described in terms of interval parameters and interval matrices. In the past two decades, a great amount of research effort has been devoted to the analysis of the interval matrix. For example, Hurwitz stability [3], [4], [5], Schur stability [6], [7], and the eigenvalue boundary problem [8], [9], [10], [11] have been studied. However, the maximum singular value of an interval matrix has not been fully investigated. Even though boundaries of singular values of an interval matrix have been studied in [12], the sign of eigenvectors was required to be unchanged with an interval perturbation. Thus, an algorithm developed in [12] was based on some restrictive assumptions. In this paper, we provide a generalized method, which can be used for finding an exact boundary of the maximum singular value of a general interval matrix.

Another unsolved problem is the power of an interval matrix at a specified order. In authors' best knowledge, there is no direct result for calculating the exact boundaries of the power of an interval matrix, which can be effectively used for calculating the impulse response bounds of a linear, time-invariant uncertain system. As some existing results for this, in [13], [10], [14], [15], convergence problems of the powers of an interval matrix were studied and it was proved that the power of an interval matrix converges to zero if the maximum spectral radius of an interval matrix is less than 1. However, the boundaries of the power of an interval matrix at a specified order has not been addressed in these publications. An useful analysis of the power of an interval matrix at a specified order can be found in [16], where it was claimed that computing the boundaries of the power of an interval matrix is an NP-hard problem. In this paper, we suggest a method for the calculation of the power of an interval matrix and it will be shown that the newly developed method can find the exact boundaries of the power of an interval matrix.

This paper consists of as follows. Section II is for the maximum singular value of an interval matrix and Section III is for the power of an interval matrix. Conclusion will be given in Section IV.

II. Maximum singular value of an interval matrix

This section consists of two different parts. In the first part (Section II-A and Section II-B), we develop algorithms for the maximum singular value of square and non-square interval matrices. In the second part (Section II-C and Section II-D), two illustrative examples are provided to demonstrate the superiority of our algorithms over an existing result.

A. Algorithm development

For our main results, we make use of Hertz's idea for finding extreme eigenvalues of a *symmetric* interval matrix [17]. In this paper, let us consider a real square non-symmetric interval matrix such as:

$$A^I = [a^I_{ij}], \quad a^I_{ij} := [\underline{a_{ij}}, \overline{a_{ij}}], i,j = 1, \dots, n \qquad (1)$$

where a^I_{ij} is an element of interval matrix A^I, $\underline{a_{ij}}$ is the lower boundary of an interval a^I_{ij}, and $\overline{a_{ij}}$ is the upper boundary of an interval a^I_{ij}. If we define the lower boundary matrix and the upper boundary matrix as $\underline{A} = [\underline{a_{ij}}]$; $\overline{A} = [\overline{a_{ij}}]$, the interval matrix can then be written as $A^I := [A^o -$

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$\Delta, A^o + \Delta]$, where the center matrix and the radius matrix are defined as

$$A^o = \frac{1}{2}(\overline{A} + \underline{A});\ \Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

A set of vertex matrices is defined as

$$A^v = [a_{ij}],\ a_{ij} \in \{\underline{a_{ij}}, \overline{a_{ij}}\},\ i,j = 1,\cdots,n.$$

In fact, the upper boundary of singular values of an interval matrix can be found as (in descending order) $\sigma_i(A^I) = \sqrt{\lambda_i((A^I)^T \otimes A^I)}$ where $\otimes$ represents multiplication of interval matrices, σ is the singular value, and λ is the eigenvalue. However, as commented in [12], the results of this method will be quite conservative. In this paper, we suggest using the following relationship between singular values and eigenvalues:

$$\begin{aligned}\sigma_i(A) &= \text{Positive}\left(\lambda_i\begin{bmatrix}0 & A^T\\ A & 0\end{bmatrix}\right)\\ &= \text{Positive}\left(\lambda_i(H)\right),\ A \in A^I \qquad (2)\end{aligned}$$

where $\text{Positive}(\cdot)$ considers only the positive part of $(\cdot)$. Obviously, H is a symmetric matrix and it is a member of the symmetric interval matrix

$$H^I = \begin{bmatrix}0 & (A^I)^T\\ A^I & 0\end{bmatrix}.$$

Hence, if we make use of the results of [17], there will be a way to find the maximum singular value of A^I. In the sequel, we briefly summarize our main idea and results.

Based on [17], since H and H^I are symmetric matrices, we have the following relationship:

$$\lambda = x^T H x = x^T \begin{bmatrix}0 & A^T\\ A & 0\end{bmatrix} x \qquad (3)$$

where x is the eigenvector corresponding to λ and $x^T x = 1$. Let us divide x into two parts such as $x^T = [y^T, z^T]$. Then, $y_i = x_i$, $i = 1,\ldots,n$ and $z_i = x_{n+i}$, $i = 1,\ldots,n$, and from (3), we obtain

$$\lambda = 2\left(\Sigma_{i=1}^n \Sigma_{j=1}^n a_{ji} y_i, z_j\right). \qquad (4)$$

Therefore, the value of λ depends on signs of y_i and z_j. That is, the maximum of λ occurs at one of vertex points of a_{ij}, which is given as:

$$a_{ij} = \begin{cases} a_{ij} = \overline{a_{ij}} & \text{if} \quad y_i z_j \geq 0\\ a_{ij} = \underline{a_{ij}} & \text{if} \quad y_i z_j < 0\end{cases} \qquad (5)$$

Now, since y and z are length-n vectors, we have a total of 2^n different sign patterns for y and 2^n different sign patterns for z. For example, when $n = 3$, sign patterns of y and z could be $+++$, $++-$, $+-+$, $+--$, $-++$, $-+-$, $--+$, $---$. In this case, we have a total of $2^3 \times 2^3 = 64$ combinations as shown in Table I and Table II.

However, Table I and Table II produce the same vertex matrices set. Therefore, in our purpose, it will be enough to check total 2^5 vertex matrices corresponding to Table I. These vertex matrices can be found easily. For example, in Table I, for the sign pattern $+--$ of y and for the sign pattern $+-+$ of z, the sign of the corresponding vertex matrix is defined by zy^T like:

TABLE I
32 SIGN PATTERNS WITH $\text{sign}(y_1) = +$ FOR 3×3 MATRIX.

y	z	y	z
$+++$	$+++$	$+-+$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$
$++-$	$+++$	$+--$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$

$$\begin{bmatrix}+\\-\\+\end{bmatrix}\begin{bmatrix}+ & - & -\end{bmatrix} = \begin{bmatrix}+ & - & -\\ - & + & +\\ + & - & -\end{bmatrix}, \qquad (6)$$

which provides the corresponding vertex matrix as follows:

$$\begin{bmatrix}\overline{a_{ij}} & \underline{a_{ij}} & \underline{a_{ij}}\\ \underline{a_{ij}} & \overline{a_{ij}} & \overline{a_{ij}}\\ \overline{a_{ij}} & \underline{a_{ij}} & \underline{a_{ij}}\end{bmatrix}. \qquad (7)$$

In the following algorithm, based on the above discussion, for a general size n square interval matrix, we can develop a generalized method:

- **Step-1**: Produce the set of ± 1 vectors with $y_1 = 1$ of length n such as
$$Y = \{y \in R^n : y_1 = 1,\ |y_j| = 1,\ \text{for } j = 2,\ldots,n\}.$$
- **Step-2**: Produce the set of ± 1 vectors of length n such as
$$Z = \{z \in R^n :\ |z_j| = 1,\ \text{for } j = 1,\ldots,n\}.$$
- **Step-3**: Make $n \times n$ diagonal matrix T_y defined by $(T_y)_{ii} = y_i$ and $(T_y)_{ij} = 0$ for $i \neq j$, $i,j = 1,\ldots,n$ where $y \in Y$.
- **Step-4**: Make $n \times n$ diagonal matrix T_z defined by $(T_z)_{ii} = z_i$ and $(T_z)_{ij} = 0$ for $i \neq j$, $i,j = 1,\ldots,n$ where $z \in Z$.
- **Step-5**: Produce a matrix set $\mathcal{S}^v := \{A_{yz} : A_{yz} = A^o + T_y \Delta T_z\ \forall\ y \in Y \ \text{ and }\ \forall\ z \in Z\}$.
- **Step-6**: Find maximum singular values of all element of the finite set $\mathcal{S}^v$ and select the largest one as the

TABLE II

32 SIGN PATTERNS WITH $\text{sign}(y_1) = -$ FOR 3×3 MATRIX.

y	z	y	z
$-++$	$+++$	$--+$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$
$-+-$	$+++$	$---$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$

maximum singular value of the interval matrix A^I.

B. Maximum singular value of non-square interval matrix

Results of the preceding section can be extended to the general non-square interval matrix case easily. Let us consider $m \times n$ interval matrix A^I. Then, H^I is $(m+n) \times (m+n)$ interval matrix. Now, introducing a length-n vector y and a length-m vector z, using the same procedure as done in the square matrix case, we have

$$\sigma(A) = 2\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{m} a_{ji} y_i, z_j\right). \tag{8}$$

Then, we can find that there is a total of 2^{m+n-1} possible combinations of vertex matrices for the maximum singular value of a non-square interval matrix. For example, for 3×2 matrix, we have total $2^3 \times 2^1$ combinations as shown in Table III.

TABLE III

16 SIGN PATTERNS FOR 3×2 NON-SQUARE MATRIX.

y	z	y	z
$++$	$+++$	$+-$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$

In Table III, for example, for the sign pattern $+-$ of y and for the sign pattern $+-+$ of z, the sign of the corresponding vertex matrix is defined by zy^T like:

$$\begin{bmatrix} + \\ - \\ + \end{bmatrix} \begin{bmatrix} + & - \end{bmatrix} = \begin{bmatrix} + & - \\ - & + \\ + & - \end{bmatrix}, \tag{9}$$

which provides the corresponding vertex matrix as follows:

$$\begin{bmatrix} \overline{a_{ij}} & \underline{a_{ij}} \\ \underline{a_{ij}} & \overline{a_{ij}} \\ \overline{a_{ij}} & \underline{a_{ij}} \end{bmatrix}. \tag{10}$$

C. Example-1: Non-square case

Let us test a non-square case. For the non-square case, we use an existing example from [12] given as

$$A^I = \begin{bmatrix} [2,3] & [1,1] \\ [0,2] & [0,1] \\ [0,1] & [2,3] \end{bmatrix}. \tag{11}$$

Using the results given in Section II-B, we find the maximum singular value of A^I as 4.54306177572459, which is very close to the value 4.543062 calculated in [12]. This result shows that our method can find the exact (without conservatism) maximum singular value of an interval matrix. Note that the suggested scheme in this paper does not require any assumption for calculating the upper boundary of the maximum singular value of interval matrix.

D. Example-2: Square case

Next, for a square matrix and to represent an exception of Deif's method [12], we create an interval matrix with the following center matrix

$$A^o = \begin{bmatrix} -3.33 & -2.24 & 0.06 \\ 1.03 & -0.34 & 1.09 \\ -2.02 & -1.02 & 2.27 \end{bmatrix} \tag{12}$$

and radius matrix

$$\Delta = \begin{bmatrix} 1.32 & 0.86 & 4.38 \\ 0.84 & 2.97 & 1.42 \\ 1.61 & 3.06 & 0.55 \end{bmatrix}. \tag{13}$$

Using the suggested method in Section II-A, we find the maximum singular value of A^I as 9.8549, but from Deif's method, we have 9.7408. For an illustrative purpose, we performed random tests. Figure 1 shows the Monte-Carlo type random tests. In this figure, the dash-dot line is the calculated maximum singular value from the suggested method (9.8549) and the solid line is the maximum singular value from Deif's method. Clearly there exist exceptions in Deif's method, while the suggested method is bounding the maximum singular value without an exception.

III. POWER OF AN INTERVAL MATRIX

In this section, we provide two main results. The first result is for an exact boundary calculation of the power of an interval matrix. In conclusion, it will be shown that exact boundaries of the power of an interval matrix can be found from the vertex matrices set. The second result is that a set of particular vertex matrices can be used for the exact

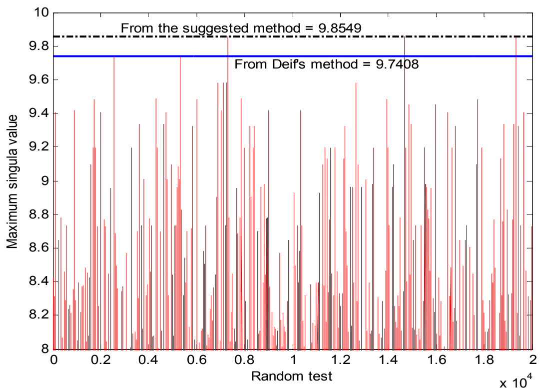

Fig. 1. Maximum singular values of randomly selected matrices and the calculated maximum singular values from the suggested method (dash-dot line) and from Deif's method (solid line).

boundary calculation of the power of an interval matrix in some special cases. Before proceeding to the main results, we need the following formal definition for the power of an interval matrix at a specified order.

Definition 3.1: The set of the power of an interval matrix is defined as

$$\mathcal{P}^k = \left\{ A^k : A \in A^I \right\},$$

where k is the order of power.

Here, we observe that A^I is an infinite set, so it seems like it is impossible to find the boundaries of $\mathcal{P}^k$. In what follows, we provide so-called sensitivity transfer method for the calculation of the boundaries of $\mathcal{P}^k$.

A. Sensitivity transfer method for power of an interval matrix

In this section, a method is developed for the calculation of the power of an interval matrix. This method computes the sensitivity of the perturbation on the nominal matrix (i.e., A) and then applies this sensitivity to the power of matrix (i.e., A^k). The set of the power of an interval matrix $\mathcal{P}^k$ defined in Definition 3.1 can be rewritten as:

$$\mathcal{A}^k = \{P^k \mid P^k = \underbrace{AAA\cdots A}_{k},\ A \in A^I\}. \quad (14)$$

Then, from the relationship $A^k = \underbrace{AA\cdots A}_{k}$, we can have

$$\frac{\partial A^k}{\partial a_{ij}} = \frac{\partial A}{\partial a_{ij}}(\underbrace{A\cdots A}_{k-1}) + A\frac{\partial A}{\partial a_{ij}}(\underbrace{A\cdots A}_{k-2}) + \cdots + (\underbrace{A\cdots A}_{k-1})\frac{\partial A}{\partial a_{ij}}. \quad (15)$$

Here, by observing that $\frac{\partial A}{\partial a_{ij}} = I_{ij}$ where I_{ij} is a matrix whose i^{th} row and j^{th} column element is 1 and the other elements are zeroes, we have

$$\frac{\partial A^k}{\partial a_{ij}} = I_{ij}(\underbrace{A\cdots A}_{k-1}) + AI_{ij}(\underbrace{A\cdots A}_{k-2}) + \cdots + (\underbrace{A\cdots A}_{k-1})I_{ij}. \quad (16)$$

Thus, we have a perturbed sensitivity (∂A^k) of A^k by amount of the uncertain change (∂a_{ij}) of a_{ij} such as

$$\partial A^k = \partial a_{ij}\left(I_{ij}(\underbrace{A\cdots A}_{k-1}) + AI_{ij}(\underbrace{A\cdots A}_{k-2} + \cdots + (\underbrace{A\cdots A}_{k-1})I_{ij}\right). \quad (17)$$

For convenience, let us use the following notation

$$\prod_{ij} := \left(I_{ij}(\underbrace{A\cdots A}_{k-1}) + AI_{ij}(\underbrace{A\cdots A}_{k-2}) + (\underbrace{A\cdots A}_{2})I_{ij} \times(\underbrace{A\cdots A}_{k-3} + \cdots + (\underbrace{A\cdots A}_{k-1})I_{ij})\right) \quad (18)$$

which simplifies (17) as $\partial A^k = \partial a_{ij}\prod_{ij}$. Hence, we find that when there is a perturbation amount of ∂a_{ij} in a_{ij}, there is a perturbation effect to A^k by amount of ∂A^k, which is related by the sensitivity transfer matrix $\prod_{ij}$. Here, noticing that each element of A perturbs A^k, we develop a method for bounding the uncertainty of A^k. Using the notation $P \in \mathcal{P}^k = \mathcal{A}^k = [\underline{P}, \overline{P}]$, we make the following proposition:

Proposition 3.1: Given the order of power k, the upper and lower boundaries associated with the elements of P occur at the power of one of the vertex matrices A^v ($A^v \in A^I$).

Proof: Let us pick arbitrary i_1 and j_1, and fix all a_{pq}, where $p, q = 1, \ldots, n$, and $p \neq i_1$ or $q \neq j_1$, to specified values such as $a_{pq} = a^*_{pq} \in [\overline{a_{pq}}, a_{pq}]$. Then, from $\partial A^k = \partial a_{i_1j_1}\prod_{i_1j_1}$, the k^{th} row and $\overline{l^{th}}$ column element of ∂A^k is determined by $\partial a_{i_1j_1}$ and $\left(\prod_{i_1j_1}\right)_{kl}$. Noticing that $\partial a_{i_1j_1} = [-\triangle a_{i_1j_1}, \triangle a_{i_1j_1}]$, the positive (negative) maximum of ∂A^k occurs at $\triangle a_{i_1j_1}$ ($-\triangle a_{i_1j_1}$) if $\left(\prod_{i_1j_1}\right)_{kl}$ is of a positive value. Otherwise, the positive (negative) maximum of ∂A^k occurs at $-\triangle a_{i_1j_1}$ ($\triangle a_{i_1j_1}$). However, the sign of $\left(\prod_{i_1j_1}\right)_{kl}$ is not determined. Hence, for arbitrary fixed i_1 and j_1, we can conclude that the positive (negative) maximum of the k^{th} row and l^{th} column element of ∂A^k occurs at one of vertex points of $a^I_{i_1j_1} = [a^o_{i_1j_1} - \triangle a_{i_1j_1}, a^o_{i_1j_1} + \triangle a_{i_1j_1}]$. Now let us pick another arbitrary i_2 and j_2. Then, due to the same reason given above, the positive (negative) maximum of the k^{th} row and l^{th} column element of ∂A^k occurs at one of vertex points of $a^I_{i_2j_2} = [a^o_{i_2j_2} - \triangle a_{i_2j_2}, a^o_{i_2j_2} + \triangle a_{i_2j_2}]$, but

$a_{i_1j_1} \in \{\underline{a_{i_1j_1}}, \overline{a_{i_1j_1}}\}$. Finally, since we can repeat above discussion to all a_{pq}, the positive (negative) maximum of the k^{th} row and l^{th} column element of ∂A^k occurs at the power of one of vertex matrices. ■

Proposition 3.1 shows that the lower and upper boundaries of the power of interval matrix can be found by checking all vertex matrices. It is important to highlight that Proposition 3.1 uses a finite set of vertex matrices for the boundary of the power of an interval matrix. This discovery is significant important, because for the first time, it was analytically proved that the power of an interval matrix can be found from the vertex matrices. However, the computational cost could be huge, because, from $\partial A^k = \partial a_{ij} \prod_{ij}$, it is required to check the whole vertex matrices of A^I to find the maximal positive or negative perturbation of elements of A^k, i.e., $(A^k)_{ij}$, $\forall A \in A^I$. Hence, in order to find the maximum and minimum of A^k, where $A \in A^I$, we have to check 2^{n^2} vertex matrices, where n is the size of the square A matrix, for each element of A^k. Thus, the total computational amount is $2^{n^2} \times 2^n = 2^{n^2+n}$. In the sequel, however, we will show that under some conditions, we do not need to check whole vertex matrices. Instead, some specified vertex matrices could be used for the calculation of the power of an interval matrix.

Proposition 3.2: If the sign of the k^{th} row and l^{th} column element of the sensitivity transfer matrix $\prod_{ij}$ does not change by ∂a_{ij}, the maximum positive and negative perturbations of the k^{th} row and l^{th} column element of $\mathcal{A}^k$, occur at the power of the following vertex matrices of A^I, respectively,

$$\begin{aligned}(A^v)|_{kl}^{+} &= \{A^v \mid A^v = [(A^o)_{ij} + s_{kl}^{ij}\triangle a_{ij}, \\ &\quad i,j = 1, \cdots, n]\} \qquad (19)\\ (A^v)|_{kl}^{-} &= \{A^v \mid A^v = [(A^o)_{ij} - s_{kl}^{ij}\triangle a_{ij}, \\ &\quad i,j = 1, \cdots, n]\} \qquad (20)\end{aligned}$$

where $s_{kl}^{ij} = \text{sign}\left(\prod_{ij}\right)_{kl}$.

Proof: From $\partial A^k = \partial a_{ij} \prod_{ij}$, the positive (negative) maximum of ∂A^k occurs at the $\triangle a_{ij}$ $(-\triangle a_{ij})$ if $\left(\prod_{ij}\right)_{kl}$ is of a positive value. Otherwise, the positive (negative) maximum of ∂A^k occurs at the $-\triangle a_{ij}$ $(\triangle a_{ij})$. This implies that the positive (negative) maximum disturbance of $(A^k)_{kl}$ occurs at $(A^o)_{ij} + s_{kl}^{ij}\triangle a_{ij}$ $((A^o)_{ij} - s_{kl}^{ij}\triangle a_{ij})$. ■

Remark 3.1: Proposition 3.2 was developed based on an assumption that the signs of $\prod_{ij}$ do not change. In the appendix, some sufficient conditions are established which can be used for checking the sign variation. However, it should be noted that if the sufficient conditions given in the appendix are not satisfied, Proposition 3.1 should be used. Hence, Proposition 3.1 and Proposition 3.2, having their own advantages and disadvantages, complement each other.

Remark 3.2: In Proposition 3.1 and Proposition 3.1, we considered a general non-symmetric square interval matrix. However, we can extend these results to a symmetric interval matrix. This work is direct by repeating (15), (16), and (17).

B. Example-1

For the usefulness of Proposition 3.2 and Lemma 5.1, let us consider the following nominal matrix, which was created using Matlab commands *rand* and *sign*:

$$A^o = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Let us suppose that there exist ± 0.001 interval uncertainties in all elements. Now, we want to find the exact upper and lower boundaries of $A^5, A \in A^I$. If we use Proposition 3.1, we need to check $2^{5^2} = 2^{25}$ vertex matrices. Indeed, in such case, the computational amount could be huge. However, from Lemma 5.1, we find that the signs of all elements of $\prod_{ij}$ do not change for all i, j. Hence, for the upper and lower boundary matrices of A^5, it is enough to use 25 vertex matrices. From these vertex matrices, we calculate the upper boundary and lower boundary matrices of A^5, $A \in A^I$, as given in $\overline{P}$ and $\underline{P}$ of the next page.

C. Example-2

Let us consider the following interval matrix:

$$A^I = \begin{bmatrix} [-1,1] & -1 & [-2,-1] \\ -1 & [0,1] & -1 \\ 1 & [1,2] & [-0.5,2] \end{bmatrix}$$

The required task is find the exact boundaries of A^5, $A \in A^I$. The conditions of Proposition 3.2 are not satisfied; so we need to use Proposition 3.1. Using this result, we find the exact boundaries of A^5, $A \in A^I$ such as:

$$\overline{P} = \begin{bmatrix} 44 & 3.375 & 44.25 \\ 5 & 44 & 27.4375 \\ 10.6875 & 31.5 & 16 \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} -11 & -36 & -31.5 \\ -25.875 & -22 & -10.6875 \\ -38.0625 & -23 & -43.7813 \end{bmatrix}$$

IV. Conclusions

In this paper, algorithms for calculating the maximum singular value of general square and non-square interval matrices have been suggested. Using the existing result [12], which was developed based on perturbation under some restrictive assumptions, we verified that the proposed method can calculate the exact maximum singular value of an interval matrix. Furthermore, from a created example, we have shown that the existing method does not find the maximum singular value in some cases while the suggested method finds the maximum singular value without an exception. As another unsolved problem in parametric interval computations, the power of an interval matrix problem has been considered in this paper also. We showed that

$$\overline{P} = \begin{bmatrix} 125.7868 & -124.2167 & -0.7030 & -68.4220 & 69.5497 \\ -130.1691 & 131.8351 & -0.6711 & 91.6420 & -90.3863 \\ 29.2777 & -28.6957 & -0.8434 & -4.7393 & 5.2731 \\ -62.3879 & 63.6182 & 11.2053 & 55.4559 & -22.5556 \\ 97.7063 & -96.2963 & 11.2414 & -40.4952 & 73.4820 \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} 124.2167 & -125.7868 & -1.2970 & -69.5800 & 68.4517 \\ -131.8351 & 130.1691 & -1.3291 & 90.3600 & -91.6163 \\ 28.7237 & -29.3057 & -1.1574 & -5.2613 & 4.7271 \\ -63.6139 & 62.3842 & 10.7953 & 54.5459 & -23.4456 \\ 96.2963 & -97.7063 & 10.7594 & -41.5052 & 72.5200 \end{bmatrix}$$

the exact boundaries of the power of an interval matrix can be found from the vertex matrices set. Furthermore, under the some special cases, a specified set of vertex matrices can be used for calculating the exact boundaries of the power of an interval matrix. Authors believe that the results introduced in this paper, due to their fundamental characteristics, can be widely and effectively used in many engineering problems such as the robust controllability test, observability test, impulse response calculation, stability analysis, etc. It is important to emphasize again that this paper presented solutions for the exact maximum singular value of an interval matrix and the power of an interval matrix, for the first time in authors' best knowledge.

V. Appendix

In this appendix, we provide sufficient conditions of Proposition 3.2. Let us write the sensitivity transfer matrix $\prod_{ij}$ such as:

$$\prod_{ij} = \sum_{p=1}^{k} A^{p-1} I_{ij} A^{k-p} \tag{21}$$

where $A \in A^I$. For convenience, let us equalize A such as $A = A^o + \Delta$ where $\Delta \in \Delta A^I = [\underline{A} - A^o, \overline{A} - A^o]$. Now, let us define $\Delta^* := \overline{A} - A^o = A^o - \underline{A}$, then we have an inequality $|\Delta| \leq \Delta^*$. Next, denoting

$$\begin{aligned} R^* \; := \; & \sum_{p=1}^{k} \big\{ \left[(|A^o| + \Delta^*)^{p-1} - |A^o|^{p-1} \right] I_{ij} \\ & \times \left[(|A^o| + \Delta^*)^{k-p} - |A^o|^{k-p} \right] \\ & + \left[(|A^o| + \Delta^*)^{p-1} - |A^o|^{p-1} \right] I_{ij} \left| (A^o)^{k-p} \right| \\ & + \left| (A^o)^{p-1} \right| I_{ij} \left[(|A^o| + \Delta^*)^{k-p} - |A^o|^{k-p} \right] \big\} \end{aligned} \tag{22}$$

and denoting $L := \left| \sum_{p=1}^{k} (A^o)^{p-1} I_{ij} (A^o)^{k-p} \right|$, we make the following lemma.

Lemma 5.1: If $L \geq R^*$ in element-wise, the signs of $\prod_{ij}$ in element-wise do not change.

Proof: See [18]. ■

When the commutative property $A^o\Delta = \Delta A^o$ holds, a less conservative condition can be derived. Note the commutative property is satisfied when A is a symmetric interval matrix, and the symmetric interval matrix system has been an important research topic as shown in [17]. For this result, we use $(A^o + \Delta)^m = \sum_{u=0}^{m} {}_mC_u (A^o)^{m-u} \Delta^u$ where ${}_mC_u = \frac{m!}{u!(m-u)!}$. Now, defining

$$\begin{aligned} S^* \; := \; & \sum_{p=1}^{k} \Big\{ \sum_{v=1}^{k-p} {}_{k-p}C_v \left| I_{ij} (A^o)^{k-v-1} \right| (\Delta^*)^v \\ & + \sum_{u=1}^{p-1} {}_{p-1}C_u (\Delta^*)^u \left| I_{ij} (A^o)^{k-u-1} \right| \\ & + \sum_{u=1}^{p-1} \sum_{v=1}^{k-p} ({}_{p-1}C_u)({}_{k-p}C_v)(\Delta^*)^{u+v} \\ & \left| I_{ij} (A^o)^{k-u-v-1} \right| \Big\} \end{aligned} \tag{23}$$

we produce the following lemma for a symmetric interval matrix.

Lemma 5.2: For a symmetric interval matrix, if $L \geq S^*$ in element-wise, the signs of $\prod_{ij}$ do not change.

Proof: See [18]. ■

The following remark is provided for some special cases.

- In Proposition 3.2, in the case of $A > 0$ element-wisely for all $A \in A^I$ or $A < 0$ element-wisely for all $A \in A^I$, the signs of $\prod_{ij}$ do not change.
- When A^I is an symmetric interval matrix and it satisfies the following property of A^I:

$$\text{sign}(AA) = \text{sign}(A) \quad \text{for} \quad \text{all} A \in A^I$$

then, the signs of $\prod_{ij}$ do not change.

References

[1] B. Ross Barmish, *New Tools for Robustness of Linear Systems*, Macmillan Publishing Company, New York, 1994.

[2] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parameter Approach*, Prentice Hall, 1995.

[3] C. I. Jiang, "Sufficient conditions for the asymptotic stability of interval matrices," *Int. J. Control*, vol. 46, pp. 1803–1810, 1987.

[4] S. J. Xu, A. Rachid, and M. Darouach, "Robustness analysis of interval matrices based on Kharitonov's theorem," *IEEE Trans. on Automatic Control*, vol. 43, no. 2, pp. 273–278, 1998.

[5] Rama K. Yedavalli, "An improved extreme point solution for checking robust stability of interval matrices with much reduced vertex set and combinatorial effort," in *2002 American Control Conference*, Arlington, VA, USA., June 25-27 2001, AACC, pp. 3902–3907.

[6] P. Batra, "On necessary conditions for real robust Schur-stability," *IEEE Trans. on Automatic Control*, vol. 48, no. 2, pp. 259–261, 2003.

[7] J. Rohn, "Positive definiteness and stability of interval matrices," *SIAM J. Matrix Anal. Appl.*, vol. 15, no. 1, pp. 175–184, 1994.
[8] A. S. Deif, "The interval eigenvalue problem," *Zeitschrift fur angewandte Mathematik und Mechanik*, vol. 71, no. 1, pp. 61–64, 1991.
[9] J. Rohn and A. S. Deif, "On the range of eigenvalues of an interval matrix," *Computing*, vol. 47, pp. 373–377, 1992.
[10] Mau-Hsiang Shih, Yung-Yih Lur, and Chin-Tzong Pang, "An ineqality for the spectral radius of an interval matrix," *Linear Algebra and Its Applications*, vol. 274, pp. 27–36, 1998.
[11] H. S. Han and J. G. Lee, "Necessary and sufficient conditions for stability of time varying discrete interval matrices," *Int. J. Control*, vol. 59, pp. 1021–1029, 1994.
[12] A. S. Deif, "Singular values of an interval matrix," *Linear Algebra and its Applications*, vol. 151, pp. 125–134, 1991.
[13] Günter Mayer, "On the convergence of powers of interval matrices," *Linear Algebra and Its Applications*, vol. 58, pp. 201–216, 1984.
[14] Chin-Tzong Pang, Yung-Yih Lur, and Sy-Ming Guu, "A new proof of Mayer's theorem," *Linear Algebra and Its Applications*, vol. 350, pp. 273–278, 2002.
[15] Sy-Ming Guu and Chin tzong Pang, "On the convergence to zero of infinite products of interval matrices," *J. Matrix Anal. Appl*, vol. 25, no. 3, pp. 739–751, 2004.
[16] Olga Kosheleva, Vladik Kreinovich, Günther Mayer, and Hung T. Nguyen, "Computing the cube of an interval matrix is NP-hard," Technical Repott UTEP-CS-04-28, Univ. of Texas El Paso, Computer Science Department, 2004.
[17] David Hertz, "The extreme eigenvalues and stability of real symmetric interval matrices," *IEEE Trans. on Automatic Control*, vol. 37, no. 4, pp. 532–535, 1992.
[18] Hyo-Sung Ahn and YangQuan Chen, "Impulse response boundary calculation based on power of interval matrix," Tech. Rep., Center for Self-Organizing and Intelligent Systems, Utah State University, 2005.