A robust Schur stability condition for interval polynomial matrix systems

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Abstract— **This paper presents a new analytical method for checking the Schur stability of interval polynomial matrices. In particular, we suggest using Markov matrices of the nominal polynomial matrix to assess its stability. The proposed technique is algebraically simple and has a relatively small computational cost, yet the result is less conservative than an existing analytical solution. The validity of the suggested method is illustrated through examples.**

I. INTRODUCTION

As commented in [1], [2], the matrix polynomial (or polynomial matrix) [3], [4] is important in the study of higher-order vector differential equations, multi-input, multi-output control systems [5], and n-D circuits. For the last two decades, the robust stability problem for polynomial matrices has been steadily studied [1], [6], [7], [8], [9], and the parametric interval concept has been widely used as shown in [10], [11], [12], [13], [14], [15]. The interval polynomial matrix system also occurs in discrete-time, multivariable problems where physical constants in the plant are subject to interval uncertainty. From a literature survey, we know that after Kharitonov [16] provided an analytical solution for the stability of continuous interval polynomials, a great amount of literature has been devoted to the study of robust stability conditions for interval matrices and interval polynomials. For more detailed robust stability analysis of interval matrices, matrix polytopes, interval polynomial matrices, and polynomial matrix polytopes, refer to [11], [12], [17], [18], [19], [13], [8], [9]. Also it has been well-known that the stability of continuous interval polynomial matrices or interval matrix polynomials can be checked by Kharitonov polynomials [6], [7], [8], [9]. However, relatively little research has been devoted to discrete interval polynomial matrices [6] or discrete interval matrix polynomials [20]. Even though recently [21] suggested an LMI condition for the robust stability of polynomial matrix polytopes and polytope type polynomial matrices, and [1] provided a stability radius for discrete polynomial matrices, as shown in the example of [21], the LMI condition was used for the polynomial matrices when coefficients of polynomial vary dependently (see Appendix of [22] for more detailed explanation about this), and in [1], it was required to calculate the norm of the inverse of the nominal polynomial matrix, which could be quite conservative. In this paper, we develop a new analytic method for checking the Schur stability of a polynomial matrix that is algebraically simple and is less conservative than the existing result [1]. In our approach, Markov matrices of the polynomial matrix are exploited by using the inverse of the nominal polynomial matrix.

The paper is organized as follows. Background materials are given in Section II and the main results are given in Section III. Examples and conclusions are given in Sections IV and V, respectively.

II. DEFINITIONS AND PRELIMINARY MATERIALS

To ensure the consistency of notations and definitions, based on [6], [7], [8], [21], [9], we begin with the definitions of interval polynomial matrices. When the (i, j)-th element of the matrix $P(z)$ is denoted by

$$p_{ij}(z) = a_{ij0} + a_{ij1}z + \cdots + a_{ijm}z^m, \ i, j = 1, \ldots, m, \quad (1)$$

where m is the degree of the polynomial, the matrix $P(z)$ is called a polynomial matrix. When the coefficients of the polynomials lie in intervals like

$$\underline{a_{ijk}} \leq a_{ijk} \leq \overline{a_{ijk}}, \ k = 0, \ldots, m, i, j = 1, \ldots, n, \quad (2)$$

where n is the dimension of the (square) matrix $P(z)$, $\overline{(\cdot)}$ is the maximum extreme value of the $(\cdot)$, and $\underline{(\cdot)}$ is the minimum extreme value of the $(\cdot)$, then these polynomial matrices are called interval polynomial matrices (denoted by $P^I(z)$) [6]. Note that polytopic polynomial matrices [11], [8] or polynomial matrix polytopes [21] should be distinguished from the generalized interval polynomial matrices.[1]

Let us consider a real monic polynomial matrix of the form:

$$P(z) = I_{m \times m}z^n + A_1 z^{n-1} + \cdots + A_{n-1}z + A_n, \quad (3)$$

where $I_{m\times m}$ is the $m \times m$ identity matrix; the coefficient matrices A_i, $i = 1, \cdots, n$ are $m \times m$ real square matrices, i.e., $A_i \in \mathbf{R}^{m\times m}$; and z is a point in complex plane, i.e., $z \in \mathbf{C}$. The following definitions are used for the stability of the polynomial matrix.

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[1] As commented in [21], these polynomial matrix polytopes are linear combinations of a set of given polynomial matrices.

Definition 2.1: [6], [8], [23] The roots λ^* of $\det(P(\lambda)) = 0$ are called eigenvalues of $P(z)$.[2] Thus, when we define a set $S_\lambda = \{\lambda \mid \det(P(\lambda)) = 0\}$, if $\max_{\lambda \in S_\lambda} |\lambda| < 1$, then the polynomial matrix $P(z)$ is robust D-stable. In this paper, robust D-stability is also called Schur stability without any notational confusion.

Definition 2.2: When elements of a matrix A are intervals such as $a_{ij} \in [\underline{a_{ij}}, \overline{a_{ij}}]$, this matrix is called an *interval matrix* A^I. The *modulus matrix* ($|A|_m$) of an interval matrix is defined as:

$$|A|_m = \left[a_{ij}^m : a_{ij}^m \in \max\{|\underline{a_{ij}}|, |\overline{a_{ij}}|, i, j = 1, \cdots, n\} \right],$$

where a_{ij}^m are elements of modulus matrix $|A|_m$. For a non-interval matrix A, we define $|A|_m = |A| = [|a_{ij}|]$, which is an absolute value of each element.

Just to emphasize, our main interest in this paper is to study the Schur stability of interval polynomial matrices $P^I(z)$. For the derivation of our main result, the following lemmas are needed.

Lemma 2.1: [5] If $P(z)$ is a real polynomial matrix and $\det(P(z))$ is not identically zero, then it is invertible (i.e., non-singular) and its inverse is a real-rational matrix.

Lemma 2.2: [27], [28] For an $m \times m$ matrix R, if $\rho(R) < 1$ (ρ means spectral radius), then $|\det(I \pm R)| > 0$.

Lemma 2.3: If $P(z)$ is invertible,[3] then $[P(z)]^{-1}$ can be expanded as $\sum_{k=0}^{\infty} T_k z^{-k}$ (i.e., $[P(z)]^{-1} = \text{span}\{Iz^{-i}, i = 0, \cdots, \infty\}$).

Proof: By Lemma 2.1, there exists $[P(z)]^{-1}$ whose elements are real-rational functions of z, denoted $p_{ij}^{-1}(z) = \sum_{k=0}^{\infty} t_{ij_k} z^{-k}$. That is, each element of $[P(z)]^{-1}$ can be expanded into its Markov parameters. Clearly then, we can write $T_k = [t_{ij_k}]$. ■

We call T_k the *Markov matrices* of $[P(z)]^{-1}$.

Lemma 2.4: [29] For any square matrices, R, T, and V, if $|R|_m \leq V$, then the following inequalities are true:

$$\rho(RT) \leq \rho(|R|_m |T|_m) \leq \rho(V |T|_m),$$

where the subscript m means the modulus matrix.

III. STABILITY CONDITION OF INTERVAL POLYNOMIAL MATRICES

The key idea of our method is to utilize Markov matrices in the region $|z| \geq 1$ of the complex plane. Our method will be developed based on the matrix determinant. We begin by rewriting the polynomial matrix (3) as:

$$\begin{aligned} P(z) &= z^{n-1}(zI + A_1 + A_2 z^{-1} + \cdots + A_n z^{-n+1}) \\ &= z^{n-1}(zI + S(z)) = z^{n-1} Q(z) \qquad (4) \end{aligned}$$

where $S(z) := A_1 + A_2 z^{-1} + A_3 z^{-2} + \cdots + A_n z^{-n+1}$, and $Q(z) := zI + S(z)$.

A. The stability of polynomial matrix

By taking the determinant of both sides of (4), we have:

$$\det[P(z)] = \det[z^{n-1} I \cdot Q(z)] = \det[z^{n-1} I] \cdot \det[Q(z)]$$

Here, observe that $z = 0$ is the solution such that $\det[z^{n-1} I] = 0$. Furthermore, $z = 0$ is not defined in $Q(z)$, because denominators become zeros. Thus, in the polynomial $Q(z)$, the complex plane without the origin is considered. Define $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Then, to determine the stability of $P(z)$ from $\det[Q(z)]$, the following lemmas can be developed:

Lemma 3.1: In $\mathbf{C}^*$, from (4), $P(z)$ is stable if and only if $Q(z)$ is stable.

Proof: In $\mathbf{C}^*$, $\det[z^{n-1} I] \neq 0$, so only z such that $\det[zI + S(z)] = 0$ makes $\det[P(z)]$ zero. Thus, from the following latent solutions:

$$S_{z^*} = \{z | \det[Q(z)] = 0, \ z \in \mathbf{C}^*\}$$

$$S_{z^{**}} = \{z | \det[P(z)] = 0, \ z \in \mathbf{C}^*\},$$

the following set equality is true: $S_{z^*} = S_{z^{**}}$. Thus, $P(z)$ is stable if and only if $Q(z)$ is stable. ■

Lemma 3.2: In $\mathbf{C}^*$, the polynomial matrix $Q(z)$ is stable if and only if $|\det[Q(z)]| > 0$, for all $|z| \geq 1$.

Proof: It is certain that, in the complex plane, there exist z such that $|\det[Q(z)]| = 0$. Thus, the condition "$|\det[Q(z)]| > 0, \ \forall \ |z| \geq 1$" is equivalent to the condition "there exist z such that $|\det[Q(z)]| = 0$ only in the disk of $|z| < 1$". Thus, by Definition 2.1, $Q(z)$ is stable. For the "only if" condition, assume that $Q(z)$ is stable. Also assume there exist any z, $|z| \geq 1$ such that $|\det[Q(z)]| \leq 0$. Then, from $S_\lambda = \{\lambda \mid \det(P(\lambda)) = 0\}$, we have $\max_{\lambda \in S_\lambda} |\lambda| \geq 1$. This contradicts the fact that $Q(z)$ is stable. Hence, $|\det[Q(z)]| > 0, \ \forall \ |z| \geq 1$ is the stability condition. ■

Therefore, based on the results of Lemma 3.1 and Lemma 3.2, we conclude that in $\mathbf{C}^*$, $P(z)$ is stable if and only if $|\det[Q(z)]| > 0$, for all $|z| \geq 1$. The following theorem is for a general case:

Theorem 3.1: If $\det(A_n) \neq 0$, then $P(z)$ is stable in $z \in \mathbf{C}$ if and only if $|\det[Q(z)]| > 0$, for all $|z| \geq 1$.

Proof: The proof can be completed by substituting $z = 0$ into $P(z)$. If $z = 0$ is substituted into $P(z)$, then $P(z = 0) = A_n$. Thus, if $\det(A_n) \neq 0$, then $z = 0$ is not a latent solution such that $\det[P(z)] = 0$. Hence, since latent solutions of $\det[P(z)] = 0$ in $\mathbf{C}$ are equivalent to latent solutions of $\det[P(z)] = 0$ in $\mathbf{C}^*$, the stability of $P(z)$ in $\mathbf{C}^*$ means the stability of $P(z)$ in $\mathbf{C}$. ■

The above theorem shows that the polynomial matrix $P(z)$ is stable if and only if $|\det[Q(z)]| > 0$, for all $|z| \geq 1$ with $\det(A_n) \neq 0$. Next, given this necessary and sufficient condition for the stability of $P(z)$, the remaining problem is to find an equivalent condition for $|\det[Q(z)]| > 0$. This will be discussed in the following subsection.

B. The stability of interval polynomial matrix

This subsection consists of two different parts. In the first part, we discuss Markov matrices of the polynomial matrix.

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[2] This notation is not unusual. For a similar discussion, refer to [24], [25], [1], [26], [4].

[3] The assumption invertible $P(z)$ is practically meaningful. From [5], we have "all polynomial matrices are invertible for almost all z unless the determinant =0 for all z." Thus, throughout the paper, we use invertible $P(z)$ as our basic assumption.

Then, based on the boundary condition of the sum of Markov matrices, we develop a robust stability condition of the interval polynomial matrices in the second part. To define Markov matrices of the polynomial matrix, we need the following lemma:

Lemma 3.3: If $\det(P(z))$ is not identically zero, then the polynomial matrix $Q(z) = zI + S(z)$ is nonsingular and $[Q(z)]^{-1}$ can be expanded using Markov matrices in $\mathbf{C}^*$ as:

$$[Q(z)]^{-1} = \sum_{k=0}^{\infty} T_k z^{n-k-1}$$

Proof: By multiplying $z^{1-n}I$ to $P(z)$, we have:

$$\begin{aligned} z^{1-n}I \cdot P(z) &= (zI + A_1 + A_2 z^{-1} + \cdots + A_n z^{-n+1}) \\ &= Q(z) \end{aligned} \tag{5}$$

Here, since $P(z)$ is nonsingular from Lemma 2.1 and $z^{1-n}I$ is nonsingular in $\mathbf{C}^*$, clearly $z^{1-n}I \cdot P(z)$ is nonsingular. Also, from Lemma 2.3, $[P(z)]^{-1} = \sum_{k=0}^{\infty} T_k z^{-k}$. Thus, the following relationship can be established easily:

$$\begin{aligned} [zI + S(z)]^{-1} &= [z^{1-n}P(z)]^{-1} \\ &= z^{n-1}[P(z)]^{-1} \\ &= \sum_{k=0}^{\infty} T_k z^{n-k-1} \end{aligned} \tag{6}$$

This completes the proof. ■

For convenience, let us write $\sum_{k=0}^{\infty} T_k z^{n-k-1}$ as:

$$\sum_{k=0}^{\infty} T_k z^{n-k-1} = \sum_{k=0}^{n-2} T_k z^{n-k-1} + \sum_{k=n-1}^{\infty} T_k z^{n-k-1}, \tag{7}$$

and replace the second term of the right-hand side by

$$\sum_{k=n-1}^{\infty} T_k z^{n-k-1} = \sum_{i=0}^{\infty} R_i z^{-i}, \tag{8}$$

where $R_0 = T_{n-1}, R_1 = T_n, R_2 = T_{n+1}, \cdots, R_i = T_{n+i-1}$.

The process for calculating Markov matrices of $Q(z)$ is summarized in the following lemma:

Lemma 3.4: If $\det(P(z))$ is not identically zero, the inverse of $Q(z)$ is expressed as $[Q(z)]^{-1} = \sum_{i=0}^{\infty} R_i z^{-i}$ in which Markov matrices are calculated by

$$R_k = -\sum_{i=1}^{k-1} A_i R_{k-i}, \ k \geq 2, \tag{9}$$

with $R_0 = 0$ and $R_1 = I$, and $A_i = 0_{m\times m}$ for $i \geq n+1$.

Proof: Due to page limitations, we omit a detailed proof. For more detail, refer to [30]. ■

In Lemma 3.4, we provided a formula for calculation of the Markov matrices of $[Q(z)]^{-1}$. Now, based on Lemma 3.4, it is easy to see that $R_k \to 0$ if and only if $Q(z)$ is stable. For more detail, let us change R_k such as

$$\begin{aligned} R_k &= -\sum_{i=1}^{k-1} A_i R_{k-i} \\ &= -A_1 R_{k-1} - A_2 R_{k-2} \cdots - A_n R_{k-n} \end{aligned} \tag{10}$$

Then, we obtain a relationship such as:

$$\begin{bmatrix} R_k \\ R_{k-1} \\ \vdots \\ R_{k-n+2} \\ R_{k-n+1} \end{bmatrix} = \begin{bmatrix} -A_1 & -A_2 & \ldots & -A_{n-1} & -A_n \\ I_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times m} & I_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \ldots & I_{m\times m} & 0_{m\times m} \end{bmatrix} \times \begin{bmatrix} R_{k-1} \\ R_{k-2} \\ \vdots \\ R_{k-n+1} \\ R_{k-n} \end{bmatrix} \tag{11}$$

Denoting equation (11) as $\overline{R}_k = \mathcal{C}\overline{R}_{k-1}$, if $\rho(\mathcal{C}) < 1$, then $\|\overline{R}_k\| \to 0$ as $k \to \infty$ which implies $\overline{R}_k \to 0$ as $k \to \infty$. Hence $R_k \to 0$ as $k \to \infty$ if and only if $\rho(\mathcal{C}) < 1$, which is an equivalent condition for the stability of $Q(z)$. Thus, if $\rho(\mathcal{C}) < 1$, we can then calculate the absolute summation of $R_k, \ k = 1, \cdots, \infty$, (denoted Σ_{R_k}) according to

$$\Sigma_{R_k} = I + \sum_{k=2}^{\infty} |R_k| \tag{12}$$

and using this summation, the following lemma can be adopted to bound $|Q(z)^{-1}|$.

Lemma 3.5: [27] In $|z| \geq 1$, the following inequality is satisfied $|Q(z)^{-1}| \leq \Sigma_{R_k}$.

In the remaining part of this section, a stability condition for interval polynomial matrices is developed. For clarity of notation, from this subsection forward the superscript o is used to denote the nominal value of any variable or parameter. In particular:

Definition 3.1: A^o denotes the nominal matrix of A^I: $A^o = \left[a^o_{ij} : a^o{}_{ij} = \frac{\underline{a_{ij}} + \overline{a_{ij}}}{2}\right]$.

Likewise, $P(z)$, $S(z)$, and $Q(z)$ defined in (3) and (4) are now denoted by $P^o(z)$, $S^o(z)$, and $Q^o(z)$, respectively.

Now, we add interval radius matrices ΔA_i to $S^o(z)$ to get the interval polynomial matrices $S^I(s)$ as follows:

$$\begin{aligned} S^I(z) &= A^o_1 + \Delta A_1 + (A^o_2 + \Delta A_2)z^{-1} + \\ &\quad \cdots + (A^o_n + \Delta A_n)z^{-n+1}, \end{aligned}$$

from which the interval coefficient matrices are defined element-wisely as $A^o_i - |\Delta A_i|_m \leq A^I_i \leq A^o_i + |\Delta A_i|_m$. We also define:

$$Q^I(z) = zI + S^o(z) + \Delta S(z), \tag{13}$$

where $\Delta S(z) = \Delta A_1 + \Delta A_2 z^{-1} + \cdots + \Delta A_n z^{-n+1}$ and define the summation of the modulus interval matrices $|\Delta A_k|_m$ as:

$$\triangle_M = \sum_{k=1}^{n} |\Delta A_k|_m. \tag{14}$$

In fact, in (12), it is not possible to estimate Σ_{R_k}, but if $Q^o(z)$ is stable, then Σ_{R_k} is bounded from (11). Let us suppose that the upper boundary of Σ_{R_k} is known as Σ^*. Then, we can find a condition for robust stability of $Q^I(z)$.[4] For this purpose, first let us rewrite $\left|\det[Q^I(z)]\right|$ using $Q^I(z) = Q^o(z)+\Delta S(z)$ such as:

$$\begin{aligned} & \left|\det\left[Q^o(z)+\Delta S(z)\right]\right| \\ = & \left|\det\left[Q^o(z)\left(I+\left(Q^o(z)\right)^{-1}\Delta S(z)\right)\right]\right| \\ = & \left|\det\left[Q^o(z)\right]\det\left[I+\left(Q^o(z)\right)^{-1}\Delta S(z)\right]\right| \\ = & \left|\det\left[Q^o(z)\right]\right|\left|\det\left[I+\left(Q^o(z)\right)^{-1}\Delta S(z)\right]\right| \quad (15) \end{aligned}$$

Then, based on Lemma 3.2 and with assumption of stable $Q^o(z)$, we have $\left|\det\left[Q^o(z)\right]\right| > 0$ at $|z| \geq 1$. Also, if $\rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) < 1$, then $\left|\det\left[I+\left(Q^o(z)\right)^{-1}\Delta S(z)\right]\right| > 0$ from Lemma 2.2. So, since $\left|\det\left[Q^o(z)\right]\right| > 0$ at $|z| \geq 1$, if $\rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) < 1$, then the interval polynomial matrix, $Q^I(z)$, is stable. Therefore, we can conclude that $Q^I(z)$ is stable if $\rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) < 1$. Next, let us investigate $\rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) < 1$. Using Lemma 2.4 and Lemma 3.5, the following inequalities are true:

$$\begin{aligned} \rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) & \leq \rho\left(\left|\left(Q^o(z)\right)^{-1}\right|_m \left|\Delta S(z)\right|_m\right) \\ & \leq \rho\left(\Sigma_{R_k}\left|\Delta S(z)\right|_m\right) \\ & \leq \rho\left(\Sigma^*\left|\Delta S(z)\right|_m\right) \quad (16) \end{aligned}$$

Furthermore, using $|z^{-k+1}|_m \leq 1$ at $|z| \geq 1$ and $k \geq 1$, the following relationships are true:

$$\begin{aligned} \left|\Delta S(z)\right|_m & = \left|\sum_{k=1}^{n}\Delta A_k z^{-k+1}\right|_m \\ & \leq \sum_{k=1}^{n}\left|\Delta A_k\right|_m\left|z^{-k+1}\right|_m \\ & \leq \sum_{k=1}^{n}\left|\Delta A_k\right|_m = \triangle_M, \end{aligned}$$

where $\triangle_M$ was defined in (14). Then, by Lemma 2.4, the following inequality is satisfied:

$$\rho\left(\left(Q^o(z)\right)^{-1}\Delta S(z)\right) \leq \rho(\Sigma^*\triangle_M)$$

Therefore, we can make the following lemma.

Lemma 3.6: If $\rho(\Sigma^*\triangle_M) < 1$, then the interval polynomial matrix, $Q^I(z)$, is stable.

Now, for the robust stability of $P^I(z)$, the following theorem is developed:

Theorem 3.2: If (i) $Q^o(z)$ is stable, (ii) $\rho(\Sigma^*\triangle_M) < 1$, and (iii) $\det(A_n^I) \neq 0$, then the interval polynomial matrices system $P^I(z)$ is robust stable.

Proof: From Lemma 3.6, if $Q^o(z)$ is stable and $\rho(\Sigma^*\triangle_M) < 1$, then $Q^I(z)$ is robust stable. Also, from Theorem 3.1, if $\det(A_n^I) \neq 0$, it is obvious that $P^I(z)$ is robust stable. ■

In the sequel, we provide a method for analytically finding Σ^*. From (11), we have

$$\begin{aligned} & |\overline{R}_{n+1}| + |\overline{R}_{n+2}| + \cdots + |\overline{R}_{n+p}| \\ = & |\mathcal{C}\overline{R}_n| + |\mathcal{C}\overline{R}_{n+1}| + \cdots + |\mathcal{C}\overline{R}_{n+p-1}| \\ = & |\mathcal{C}\overline{R}_n| + |\mathcal{C}^2\overline{R}_n| + \cdots + |\mathcal{C}^p\overline{R}_n| \\ \leq & \left(|\mathcal{C}| + |\mathcal{C}^2| + \cdots + |\mathcal{C}^p|\right)|\overline{R}_n| \quad (17) \end{aligned}$$

where we used the inequality $|AB| \leq |A||B|$. Here, if $\mathcal{C}$ is diagonalizable such as $\mathcal{C} = \mathcal{X}\Lambda\mathcal{X}^{-1}$, we then change (17) as follows:

$$\begin{aligned} & \left(|\mathcal{C}| + |\mathcal{C}^2| + \cdots + |\mathcal{C}^p|\right)|\overline{R}_n| \\ = & \left(|\mathcal{X}\Lambda\mathcal{X}^{-1}| + |\mathcal{X}\Lambda^2\mathcal{X}^{-1}| + \cdots + |\mathcal{X}\Lambda^p\mathcal{X}^{-1}|\right)|\overline{R}_n| \\ \leq & |\mathcal{X}|\left(|\Lambda| + |\Lambda^2| + \cdots + |\Lambda^p|\right)|\mathcal{X}^{-1}||\overline{R}_n| \quad (18) \end{aligned}$$

which yields the following general formula:

$$\sum_{i=1}^{p}|\overline{R}_{n+i}| \leq |\mathcal{X}|\left(\sum_{i=1}^{p}|\Lambda^i|\right)|\mathcal{X}^{-1}||\overline{R}_n|. \quad (19)$$

Now taking $p \to \infty$, if $Q^o(z)$ is stable, then we have

$$\begin{aligned} \lim_{p\to\infty}\sum_{i=1}^{p}|\overline{R}_{n+i}| & = |\mathcal{X}|\lim_{p\to\infty}\left(\sum_{i=1}^{p}|\Lambda^i|\right)|\mathcal{X}^{-1}||\overline{R}_n| \\ & \leq |\mathcal{X}|\mathrm{diag}\left(\frac{|\lambda_l|}{1-|\lambda_l|}\right)|\mathcal{X}^{-1}||\overline{R}_n| \end{aligned} \quad (20)$$

where $\mathrm{diag}(\cdot)$ is a diagonal matrix composed of diagonal terms $(\cdot)$. Since $\mathcal{X}$, λ_l, $\mathcal{X}^{-1}$, and $\overline{R}_n$ are known, we can estimate the boundary of $\lim_{p\to\infty}\sum_{i=1}^{p}|\overline{R}_{n+i}|$. Now denoting $\mathcal{T} := |\mathcal{X}|\mathrm{diag}\left(\frac{|\lambda_l|}{1-|\lambda_l|}\right)|\mathcal{X}^{-1}|$ and $\mathcal{F}_n := \mathcal{T}|\overline{R}_n|$, and taking first m rows of $\mathcal{F}_n$, which is denoted as $\mathcal{D}_n$ (i.e., $\mathcal{D}_n := \mathcal{F}_n(1:m, 1:m)$), we have the following inequality:

$$\Sigma_{R_k} \leq \sum_{i=1}^{n}|R_i| + \mathcal{D}_n \quad (21)$$

Therefore, since $\sum_{i=1}^{n}|R_i|$ and $\mathcal{D}_n$ are calculated, we can analytically estimate the upper boundary of Σ_{R_k}. However, (21) could be conservative. To accurately estimate the upper boundary of Σ_{R_k}, we introduce an operator $\mathcal{F}_q := \mathcal{T}|\overline{R}_q|$, where $q \gg n$ and denoting $\mathcal{D}_q := \mathcal{F}_q(1:m, 1:m)$, we have a more accurate upper boundary of Σ_{R_k} such as

$$\Sigma_{R_k} \leq \sum_{i=1}^{q}|R_i| + \mathcal{D}_q := \Sigma^*. \quad (22)$$

Now, the above argument (accurate upper boundary of Σ_{R_k}) is summarized in the following theorem:

Theorem 3.3: If $Q^o(z)$ is stable, in (22), as $q \to \infty$, $\mathcal{D}_q \to 0$; hence $\Sigma^* \to \Sigma_{R_k}$.

Proof: Since $\mathcal{T}$ is fixed, from (11), $|\overline{R}_q| \to 0$ as $q \to \infty$ if and only if $\rho(\mathcal{C}) < 1$. Therefore, since $|\overline{R}_q| = 0$ if and only if $\overline{R}_q = 0$, the proof is immediate. ■

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[4] Note: similar discussions for this can be found in [27], [28]. In this paper, we reformulated an idea of [27] into the interval polynomial matrices system.

Theorem 3.3 shows that we can estimate an accurate upper boundary of Σ_{R_k} (i.e., Σ^*) by taking very large q in (22).

IV. EXAMPLES

In this section, we test the suggested algorithm with respect to conservatism. From the existing literature, however, it is difficult to find a prototype example to use. Most of examples we have found consider the continuous case, e.g., [6], [7], [8], [21], [9]. Further, though example 3.5 of [1] is discrete, it is a special case which allows the analytical calculation of $P(\lambda)^{-1}$ and it is very difficult to calculate $P(\lambda)^{-1}$ analytically in general case. Thus, in this paper, for the comparison purpose we use example 4.3 of [1], which uses a numerical range for $\|P(\lambda)^{-1}\|_2$.

A. Example-1

The main disadvantage of the suggested method in [1] is to calculate $\|P(\lambda)^{-1}\|_2$ analytically. Although [1] provides a method for this, the method is quite complicated and the result could be very conservative, because it numerical ranges (see Theorem 4.1 of [1]). In example 4.3 of [1], the stability radius of a polynomial matrix was calculated under some conditions. Let us use the example of [1], given as:

$$P^o(z) = Iz^3 + A_1^o z^2 + A_2^o z + A_3^o, \tag{23}$$

where the coefficient matrices are Hermitian, and satisfy the conditions $0 \leq \lambda_{min}(A_1^o) \leq \lambda_{max}(A_1^o) \leq 1/3$, $-1/9 \leq \lambda_{min}(A_2^o) \leq \lambda_{max}(A_2^o) \leq 1/9$, and $-1/27 \leq \lambda_{min}(A_3^o) \leq \lambda_{max}(A_3^o) \leq 1/9$. Since [1] does not provide the coefficient matrices, in order to satisfy the above conditions, we simply selected matrices such as:

$$A_1^o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

$$A_2^o = \begin{bmatrix} -0.111 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.111 \end{bmatrix}$$

$$A_3^o = \begin{bmatrix} -0.0370 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1111 \end{bmatrix}$$

In [1], the analytical perturbation radius of $P^o(z)$ is calculated as 0.0141, i.e., $\|[0_{3\times 3} \;\; \Delta_1 \;\; \Delta_2 \;\; \Delta_3]\|_2$ could be 0.0141. Thus, as far as $\|[0_{3\times 3} \;\; \Delta_1 \;\; \Delta_2 \;\; \Delta_3]\|_2 \leq 0.0141$, the polynomial matrices system is robust stable. Let us use our method for this analytical stability radius. From the companion form $\mathcal{C}$, we find that the nominal system is stable and has different eigenvalues. From Theorem 3.3 and (22), selecting $q = 100$, we found that

$$\Sigma^* = \sum_{i=1}^{q} |R_i| + \mathcal{D}_q = \begin{bmatrix} 1.1739 & 0 & 0 \\ 0 & 1.1152 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Also, for the uncertainty, we provided 10 percent intervals to A_1^o, A_2^o, and A_3^o, from which $\|[0_{3\times 3} \;\; \Delta_1 \;\; \Delta_2 \;\; \Delta_3]\|_2 = 0.0369$. From (14), we calculated

$$\triangle_M = \begin{bmatrix} 0.0148 & 0 & 0 \\ 0 & 0.011 & 0 \\ 0 & 0 & 0.0556 \end{bmatrix}$$

Finally, using the calculated $\sum_{i=1}^{q} |R_i| + \mathcal{D}_q$ and $\triangle_M$, we found that $\rho(\Sigma^* \triangle_M) = 0.0833$; hence the interval polynomial system is robust stable with 10 percent uncertainty, which cannot be concluded in [1].

B. Example-2

Let us first consider the following general non-symmetric polynomial matrix:

$$P^o(z) = I_{2\times 2} z^3 + A_1^o z^2 + A_2^o z + A_3^o, \tag{24}$$

where the coefficient matrices are given as:

$$A_1^o = \begin{bmatrix} 0.4 & -0.3 \\ 0.4 & 0.1 \end{bmatrix}; \; A_2^o = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & -0.5 \end{bmatrix}$$

$$A_3^o = \begin{bmatrix} 0.0 & 0.5 \\ -0.1 & 0.15 \end{bmatrix}$$

From the corresponding $\mathcal{C}$ matrix, the eigenvalues are calculated as: $-0.0535{+}0.8125i$; $-0.0535{-}0.8125i$; -0.8696; -0.5086; 0.7612; 0.2240. Thus, the nominal system is stable. Also, since $\det(A_3^I)$ is not zero, the suggested method can be used. It is assumed that there can exist element-wise interval uncertainty in the nominal matrices such as

$$\Delta A_1 = \begin{bmatrix} 0.0432 & 0.0324 \\ 0.0432 & 0.0108 \end{bmatrix}; \; \Delta A_2 = \begin{bmatrix} 0.0324 & 0.0324 \\ 0.0432 & 0.0540 \end{bmatrix}$$

$$\Delta A_3 = \begin{bmatrix} 0.0 & 0.0540 \\ 0.0108 & 0.0162 \end{bmatrix}$$

To apply Theorem 3.3, we selected $q = 50$, from which we found

$$\Sigma^* = \sum_{i=1}^{50} |R_i| + \mathcal{D}_50 = \begin{bmatrix} 3.1502 & 1.8140 \\ 1.5848 & 4.1860 \end{bmatrix}.$$

Using these matrices, we found $\rho(\Sigma^* \triangle_M) = 0.9979$ which shows that the system is robust stable (almost marginal stable). For the comparison purpose, we used Theorem 3.1 of [1]. It is, however, difficult to find the infimum of $\frac{1}{\sqrt{\sum_{k\in J} |\lambda|^{2k} \|P(\lambda)^{-1}\|_2}}$ for all $\lambda \in \partial\Omega$ (for notation, refer to [1]); hence we performed a random simulation test and found that $\lambda = -1$ is the best (this is not analytical solution, instead we did 2000 random tests to find the best λ). Using $\lambda = -1$, we calculated $\frac{1}{\sqrt{\sum_{k\in J} |\lambda|^{2k} \|P(\lambda)^{-1}\|_2}} = 0.1175$, and calculated $\|[\Delta_0 \;\; \Delta_1 \;\; \Delta_2 \;\; \Delta_3]\| = 0.1174$ which is also almost marginal stable. Hence, from this test, we found that when the exact minimum of $\frac{1}{\sqrt{\sum_{k\in J} |\lambda|^{2k} \|P(\lambda)^{-1}\|_2}}$ of Theorem 3.1 of [1] is found, the stability condition of [1] is almost equal to the stability condition of our method. However, as commented in [1], it is tough to find the exact minimum of $\frac{1}{\sqrt{\sum_{k\in J} |\lambda|^{2k} \|P(\lambda)^{-1}\|_2}}$ (so they used numerical range for an approximation, but the result is quite conservative as shown in example-1 of this paper).

V. CONCLUSION

In this paper, a new method for checking the Schur stability of interval polynomial matrices system was suggested. Our method checks the stability in a simple manner and the derivation process is analytically reasonable. From a comparison with an existing method, we have found that our method is less conservative as demonstrated in example-1 and computationally very simple. Furthermore, we have found that our method provides almost the same stability condition as [1] when the exact minimum of Theorem 3.1 of [1] is found.[5]

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[5] However, actually it is very tough to find this minimum, so [1] developed Theorem 4.1, which results in a conservative stability radius.