Local Analysis of Long Range Dependence Based on Fractional Fourier Transform

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Abstract—**The long range dependence (LRD) of stationary process is characterized by the Hurst parameter. In practice, previous methods for estimation of the Hurst parameter might have poor performance when processing the non-stationary time series or trying to distinguish the slight difference between very long stochastic processes. This paper explores the use of fractional Fourier transform (FrFT) for estimating the Hurst parameter. The time series was processed locally to achieve a reliable local estimation of the Hurst parameter. The biocorrosion signal which is very popular in biological engineering was studied as an example to show the long range dependence properties. After comparing with the commonly used wavelet based method and another method based on Matlab's polyfit, the new Hurst parameter estimator proposed in this paper is proved to be more robust for non-stationarity and can show the slight difference clearly between those very long sets of biocorrosion data.**

Index Terms—**biocorrosion signal, fractional Fourier transform, Hurst parameter, long range dependence, parameter estimation**

I. INTRODUCTION

THE first model for long range dependence was introduced by Mandelbrot and Van Ness (1968) in terms of fractional Brownian motions, and has since been extensively developed. Long-range dependent processes are characterized by their auto-covariance functions. In LRD processes there is a strong coupling between values at different times. This indicates that the decay of the auto-covariance function is hyperbolic and slower than exponential decay, and that the area under the function curve is infinite. Consider a second order stationary time series $Y = Y\{(k)\}$ with zero mean as an example. The time series Y is said to be long-range dependent provided its auto-covarianc $r_Y(k) = Cov(Y(k), Y(0)) = E(Y(k)Y(0))$ decays slowly as a function of the lag k, so that the series $\sum_k r_Y(k)$ is not summable [1,2].

The Hurst parameter H characterizes the degree of LRD. The LRD is typically modeled by supposing a power law decay of the spectral density, H equals to $(1+\alpha)/2$ in which α is the power. A process is said to have long range dependence when $0.5 < H < 1$ [2]. Many methods have been proposed for Hurst parameter estimation like wavelet-based [3], local Whittle [4], R/S analysis [5], periodogram methods [6] and so on. On the other hand, Stoev, Taqqu, Park and Marron (2004) have shown that non-stationarity effects such as abrupt shifts in the mean and some other contaminations may affect the above methods and result in overestimating the Hurst parameter [7]. The goal of this paper is to devise an algorithm that is robust to those effects. We propose to use Fractional Fourier Transform (FrFT) for Hurst parameter estimation. FrFT has a computational complexity proportional to the fast Fourier transform algorithm. In the long range dependence applications, it is possible to improve performance by the use of the FrFT [8]. Since the fractional Fourier transform can be computed in about the same amount of time as the ordinary Fourier transform, these performance improvements come without additional cost. Besides, FrFT has a strong relationship with wavelet transform which is very suitable for analyzing LRD [9]. This will be discussed in detail in the following sections.

The FrFT based Hurst estimation method in this paper implements a set of windows over the spectrum to process the data locally for some time series long enough to have partial non-stationarity. Stilian Stove (2004) has already performed local analysis of self-similar data series based on wavelet spectrum. Local deviation from long range dependence could be visualized by the local analysis.

According to the research in recent years, financial data and communications networks data [10] can exhibit long range dependence. In this paper we first explore the long range dependence in biocorrosion data. In the field of biomedical engineering, electrochemistry is applied to deal basically with behavior of implant devices in terms of corrosion. Today, medical implant devices in the body cover a range of materials and applications. Biocorrosion, as an important interfacial process of the materials, is defined as a chemical or electrochemical reaction between a material, usually a metal, and its biological environment that produces a deterioration of the metal and its properties [11]. Different types of microbes are used in microbiologically influenced corrosion (MIC) such as sulphur reducing bacteria, sulphur oxidizing bacteria, organic acid producing bacteria, iron oxidizing bacteria, etc [12]. Metals which are affected by MIC include stainless steel, titanium, gold, etc. In this paper, we utilized stainless steel as electrodes and tested them in different artificial saliva to get

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the desired biocorrosion data. An example of biocorrosion potential noise is shown in Fig. 1.

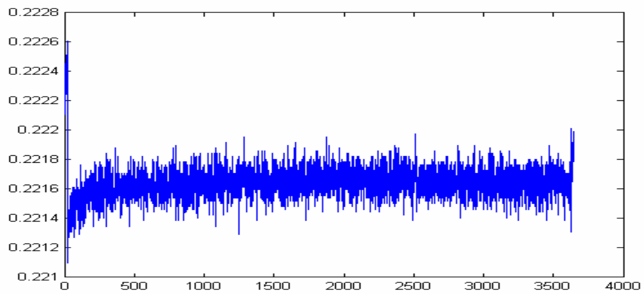


Fig. 1. An example of biocorrosion potential noise measurement with x-axis the time in 'second' and y-axis the potential in 'Volt'.

The fluctuations in Fig. 1 may suggest long range dependence [13]. For a stationary signal repeating into infinity with the same periodicity, one can see that the random fluctuation in Fig. 1 may possess non-stationarity in some parts when the time series becomes very long, like the 24 hour biocorrosion data used in this paper.

The paper is organized as follows. In Section II**,** we derive the relationship between the FrFT and the wavelet transform, and deduce the Hurst parameter estimation from FrFT spectrum. Section III discusses the local analysis for estimating the Hurst parameter. Section IV analyzes the effect of non-stationarity. Section V presents some of the experimental results of biocorrosion data and compares them with results of the other two methods to illustrate the benefits offered by the proposed FrFT based local analysis method. Conclusions are drawn in Section VI. All the numerical experiments were implemented in Matlab.

II. HURST PARAMETER ESTIMATION BASED ON FrFT

Consider a second order stationary stochastic signal $Y = \{Y(t)\}$. $Y(t)$ is said to be long-range dependent if it can be modeled by a power-like decay of the covariance:

$$r_Y(k) = EY(k)Y(0) \leftrightarrow c_Y |k|^{-\gamma}, k \to \infty, 0 < \gamma < 1 \tag{1}$$

where '$\leftrightarrow$'means the ratio of the left and the right hand sides converges to 1. Imposing the condition (1) on the spectral density f_Y of Y, as $\xi \to \infty$ we get,

$$f_Y(\xi) \leftrightarrow c_f |\xi|^{-\alpha}, 0 < \alpha < 1, \tag{2}$$

where $c_f > 0$ and $f_Y(\xi) = (2\pi)^{-\frac{1}{2}} \sum_{k \in Z} e^{i\xi k} r_Y(k)$.

Since the Hurst parameter is related with α by $H = (1+\alpha)/2$ and α can be estimated by the spectral density f_Y according to (2), the Hurst parameter H can be estimated according to f_Y. Therefore, the relationship between FrFT spectrum and classical spectral density f_Y can be used to estimate the Hurst parameter.

The definition of FrFT [14] of a signal $f_a(\xi)$ is

$$f_a(\xi) = \int_{-\infty}^{\infty} K_a(\xi, u) f(u) du \tag{3}$$

where,

$$K_a(\xi, u) = A_\alpha \exp[i\pi(\cot \alpha \xi^2 - 2\csc \alpha \xi u + \cot \alpha u^2)],$$
$$A_\alpha \equiv \sqrt{1 - i\cot \alpha}, \alpha \equiv \frac{a\pi}{2}.$$

Making the change of variable by $x = \xi \sec \alpha$ and denoting the left hand side of (3) by $g(x) = f_a(x/\sec \alpha)$, we can obtain the following result [15].

$$g(x) = C(\alpha) \exp[-i\pi x^2 \sin^2 \alpha] \int_{-\infty}^{\infty} \exp[i\pi(\frac{x-u}{\tan^{1/2}\alpha})^2] f(u) du \tag{4}$$

where $C(\alpha)$ is a constant that depends only on α.

Note that equation (4) has certain similar characteristics of a wavelet transform. The continuous wavelet transform can be defined as [16],

$$WT_f(x,a) = \int f(u) \psi^*_{x,a}(u) du, \ \psi_{\tau,a}(u) = \frac{1}{|a|^{1/2}} \psi(\frac{u-x}{a}) \tag{5}$$

where $\tau, a, \psi(t), \psi_{\tau,a}(t)$ respectively show scan time, scale, mother wave, wavelet.

Let the mother wave $\psi(t) = \exp(i\pi t^2)$, the wavelet transform in equation (5) can be changed to

$$WT_f(x,a) = \frac{1}{|a|^{1/2}} \int \exp(i\pi(u-x)^2 / a^2) f(u) du. \tag{6}$$

Clearly, equation (6) has the same form as FrFT in (4) except the scaling factor. Let $a = \tan^{1/2} \alpha$. Equation (6) becomes

$$WT_f(x,a) = \frac{1}{|\tan(\alpha)|^{1/4}} \int \exp[i\pi\left(\frac{x-u}{\tan^{1/2}\alpha}\right)^2] f(u) du$$
$$= \frac{\exp[i\pi x^2 \sin^2 \alpha]}{C(\alpha)|\tan(\alpha)|^{1/4}} g(x) \tag{7}$$

which establishes the relationship between the FrFT and the wavelet transform.

The stationarity of Y implies the stationarity of the wavelet coefficients $\{d_{j,a}(Y)\}_{a \in z}$, for all scales $j \in Z$. Let ξ_j denote the mean energy of the wavelet coefficients at the scale j, that is,

$$\xi_j = Ed^2_{j,a}(Y). \quad (8)$$

By Mallat's algorithm [2], one can obtain a triangular array of approximate wavelet coefficients. Thus, one can estimate $\log_2(\xi_j)$ by using the sample energy of these coefficients:

$$S_j(Y) = WT_f(j,a) = \log_2(\frac{1}{N_j}\sum_{k=1}^{N_x} d^2_{j,a}(Y))$$
$$\approx \log_2(\xi_j) = \log_2(Ed^2_{j,a}(Y)). \quad (9)$$

According to (7), substituting $g(x)$ into equation (9), we get the FrFT spectrum

$$G_j(Y) = g(x) = \frac{C(\alpha)|\tan(\alpha)|^{1/4}}{\exp[i\pi j^2 \sin^2\alpha]}\log_2(Ed^2_{j,a}(Y)). \quad (10)$$

By using the Parseval identity and a change of variables, it can be shown that

$$Ed^2_{j,a}(Y) = \int_r \psi_{j,a}(t)\int_R \psi_{j,a}(s) r_Y(t-s)dsdt = \int_R |\hat{\psi}_{j,a}(\xi)|^2 f_Y(\xi)d\xi$$

$$= 2^j\int_R |\hat{\psi}(2^j\xi)|^2 f_Y(\xi)d\xi = \int_R |\hat{\psi}(\eta)|^2 f_Y(\eta/2^j)d\eta \quad (11)$$

where $\hat{\psi}(\xi) = \sqrt{2\pi}\int_R e^{i\xi t}\psi(t)dt$ denotes the Fourier transform of the function ψ. The final expression in (4) relates the mean energy ξ_j of the wavelet coefficient $d_{j,a}(Y)$ to the spectral density of the stationary signal $Y(t)$. For large scales, j, the function $f_Y(\eta/2^j), \eta \in R$ can be viewed as a zoomed version of the spectral density $f_Y(\eta)$ around the zero frequencies. Therefore, the integral in the right hand side of (11) picks out the spectral behavior of Y at low frequencies.

Substituting (2) into (11) yields, as $j \to \infty$,

$$Ed^2_{j,a}(Y) \leftrightarrow c_f\int_R |\hat{\psi}(\eta)|^2 |\eta/2^j|^{-\alpha} d\eta = c_f C 2^{j\alpha} \quad (12)$$

where $C = C(\psi,\alpha) = \int_R |\hat{\psi}(\eta)|^2 |\eta|^{-\alpha} d\eta$.

According to $H = (1+\alpha)/2$ and equations (10) and (12), the Hurst parameter can be estimated using the fact that

$$G_j(Y) \leftrightarrow (2H-1)j + const. \quad (13)$$

By using a linear regression of the FrFT spectrum $G(j)$ on the scales j, where $1 < j_1 < j_2 < J$, the Hurst parameter H of f can be estimated as,

$$\hat{H}_{[j1,j2]} = \frac{1}{2}\sum_{j=j_1}^{j_2}\omega_j g_j(Y) + \frac{1}{2} \quad (14)$$

where ω_j are such that $\sum_{j=j_1}^{j_2}\omega_j = 0$ and $\sum_{j=j_1}^{j_2} j\omega_j = 1$.

III. LOCAL ANALYSIS

Having established the FrFT based Hurst parameter estimation method, let us now consider $\hat{H}_{[j,j+1]}$ given by

$$\hat{H}_{[j,j+1]} = \frac{(G_{j+1}(Y) - G_j(Y))}{2} + 0.5, j = 1,2,...,J-1. \quad (15)$$

In (15), $G_{j+1}(Y) - G_j(Y)$ is the local slope of the wavelet spectrum. From equation (13), $\hat{H}_{[j,j+1]}$ should be very close to H at large scale j. Therefore, $\hat{H}_{[j_1,j_2]}$ in (14) can be expressed as

$$\hat{H}_{[j_1,j_2]} = \sum_{j=j_1}^{j_2-1} v_j \hat{H}_{[j,j+1]} \quad (16)$$

where $v_j = \omega_{j+1} + ... + \omega_{j_2}$.

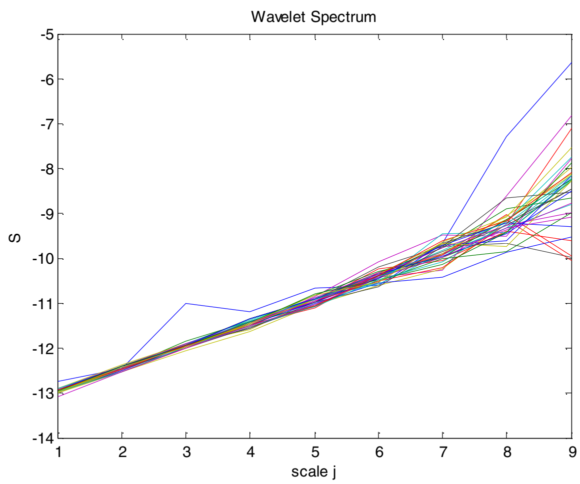
For Y, choose the initial window size $\omega < N$ and divide the time series Y into $[N = \omega]$ non-overlapping time series $Y_r, r = 1,...,[N = \omega]$, where Y_r is the time series corresponding to the rth window. The first $[N = \omega] - 1$ windows are of size ω and the last one is of size $N - \omega([N = \omega] - 1) \geq \omega$. Compute the wavelet spectrum of the time series Y within each window and obtain a matrix G of dimensions $(J \times [N = \omega])$, where $J = J(\omega) < \log_2(\omega)$ equals the number of available dyadic scales in each of the windows. The (j,r)th element of the matrix G is defined as

$$G_j(r) = \log_2(\frac{1}{N_j}\sum_{k=1}^{N_j} g^2_{j,a}(Y_r)) \quad (17)$$

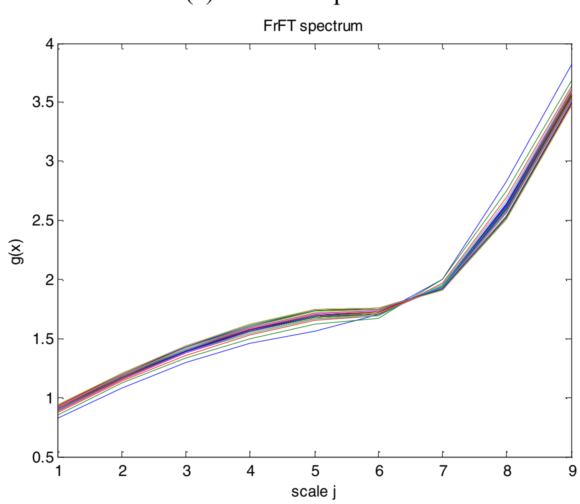
where $g_{j,a}(Y_r)$ is the FrFT spectrum coefficient of time series Y_r in the r-th window.

IV. EFFECT OF NON-STATIONARITY

Most of the Hurst parameter estimation methods are based on the assumption that the time series is stationary [17]. Non-stationarity will result in poor performance or even failure of the estimation [7]. Stove, et al (2004) have shown that non-stationarity effects such as abrupt shifts in the mean would yield a steep wavelet spectrum and overestimate the Hurst parameter. The two sub-plots in Fig. 2 are wavelet spectrum and FrFT spectrum of the biocorrosion potential noise of the stainless steel electrode with bacteria in an artificial saliva called Jenkin's Solution. The corrosion electrode is put in the solution for 24 hours in room temperature. The Zero Resistance Ammeter test is performed for electrochemical noise (ECN) measurement. The sampling period is 0.5 second.



(a) Wavelet spectrum



(b) FrFT spectrum

Fig. 2. Local wavelet & FrFT spectra of the biocorrosion noise

Fig. 2. (a) shows that the wavelet spectrum is consistent with the long-range dependence. At the large scales, there is some variability in the spectrum which indicates some inconsistency. This may affect the global analysis of long range dependence for global Hurst parameter. The FrFT spectrum in Fig. 2. (b) is, however, more robust on large scales compared to the wavelet spectrum.

V. EXPERIMENTAL RESULTS AND ANALYSIS

A. Local FrFT Based Estimator vs. Local Wavelet Based Estimator

First, we apply the FrFT based Hurst parameter estimator to the 24 hour biocorrosion data. When implementing the method, it is important to choose the right starting scale j_1. This is because at large scales the long-range dependent time series becomes self-similar, while at small scales the FrFT spectrum may not have the same slope. This paper used the graphical tool [3] for choosing the range of scales. At the beginning, we set $j_1 = 5$ and $j_2 = J(\omega)$ for each window, and then estimate the local Hurst parameter $\hat{H}(r)$.

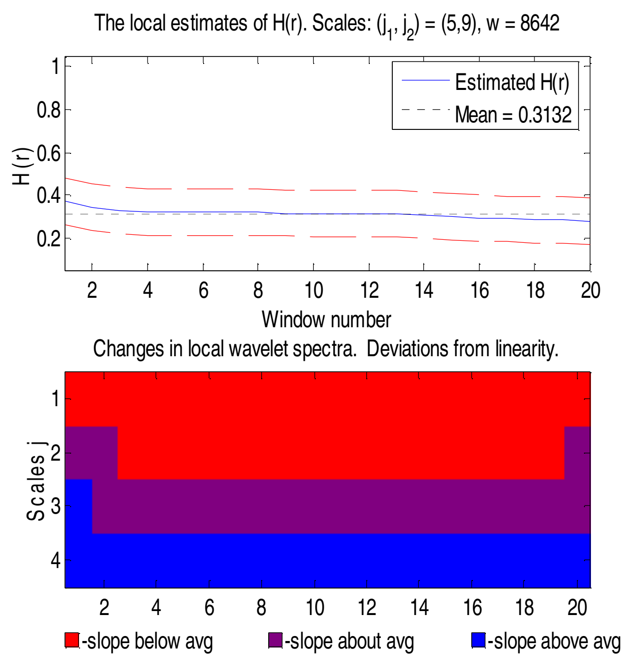


Fig. 3. Local estimation of Hurst parameter by FrFT spectrum. Applied to biocorrosion potential noise of stainless steel electrode in Jenkin's Solution for 24 hours.

In Fig. 3, the top plot is the local $H(r)$ estimate with the mean and a band of their 95% confidence intervals while the bottom plot visualizes the deviations in the local FrFT spectrum [1]. The presence of red (white) and blue (black) indicates that the choice of scales j_1, j_2 is not suitable and these Hurst parameter estimates may not be meaningful. The slope of the FrFT below the average is too much.

According to the importance of choosing the scale j_1 and the results shown in Fig. 3, the choice of j_1 should be changed. After checking the FrFT estimator with several j_1, we found out a proper starting scale $j_1 = 7$, which gives a good estimation in Fig. 4. The bottom plot of Fig. 4 shows that there are fewer red (white) or blue (black) colors than those of Fig.3 which means the local Hurst parameter estimations in Fig. 4 is more reliable than those in Fig. 3. The top plot in Fig. 4 gives the mean value of Hurst parameter which is 0.77291, right between 0.5 and 1. It indicates that the 24 hour biocorrosion data have long range dependence.

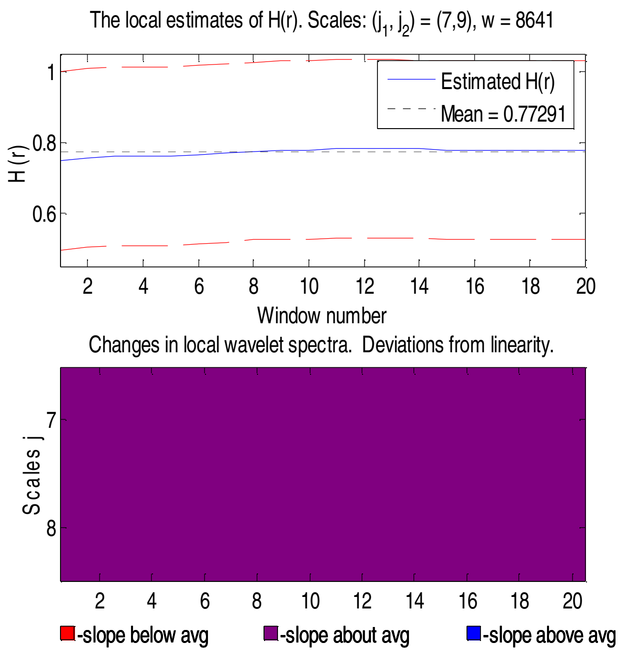


Fig. 4. With modified starting scale j_1, local estimation of Hurst parameter by FrFT spectrum. Applied to biocorrosion potential noise of stainless steel electrode in Jenkin's Solution for 24 hours.

Now, let us apply the wavelet spectrum based Hurst parameter estimation method for a comparison. In case the data have non-stationarity, we also utilize local analysis for the wavelet based estimator. The same biocorrosion data was tested. The top plot in Fig. 5 shows the estimates of the local Hurst parameters, their mean value and their 95% confidence intervals over the window number parameter r. The bottom plot is the visualization of deviations. The starting scale is $j_1 = 7$. As shown in the figure, the wavelet estimator mainly has two disadvantages. First, the bottom plot of wavelet estimator has more red (white) and blue (black) patches, which indicates the bigger deviation and misleading Hurst estimates. Second, both the subplots in Fig. 5 show some non-starionary effects. For example, for window numbers r = 3, 8, 14 and 19, there are big fluctuations in the estimates in the top plot. Correspondingly, the red (white) and blue (black) patches in the bottom plot of Fig. 5 also appear at those window numbers. On the contrary, the subplots in Fig. 4 show smooth estimations which indicates the FrFT based estimator is more accurate and more robust to non-stationarity.

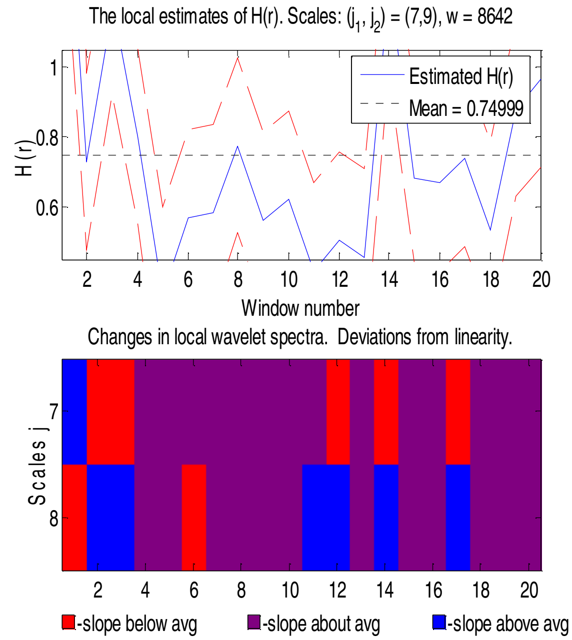


Fig. 5. Local estimation of Hurst parameter by wavelet spectrum. Applied to biocorrosion potential noise of stainless steel electrode in Jenkin's Solution for 24 hours.

B. *Global Estimator vs. Local FrFT Based Estimator*

In what follows, the table of the results of Hurst parameter estimation was given out by the conventional method using Matlab's ployfit [18]. It estimates the Hurst parameter on full scale, globally.

TABLE I
HURST PARAMETERS OF BIOCORROSION SIGNALS OF STAINLESS STEEL ELECTRODE WITH BACTERIA IN THREE DIFFERENT ARTIFICIAL SALIVA

Hurst parameter	5 min	30 min	24 hours
Solution A	0.82	0.99	1.00
Solution B	1.02	0.98	1.00
Solution C	0.85	0.94	1.00

The corrosion potential of Stainless Steel electrode in three different simulated saliva solutions, namely: (1) Jenkin's Solution, (2) Tomasi's Solution, and (3) NaCl solution are determined via electrochemical noise (ECN) technique. As shown in Table I, the estimated Hurst parameters vary for three different solutions in five minutes and 30 minutes tests, respectively. But for the 24 hours test, the Hurst parameters of the biocorrosion data in three solutions have the same value 1.00 which is not realistic, so the technique failed to tell the differences of the Hurst parameters between those very long time series.

Now, let us apply the local FrFT based estimator to biocorrosion data of 3 solutions in 24 hours.

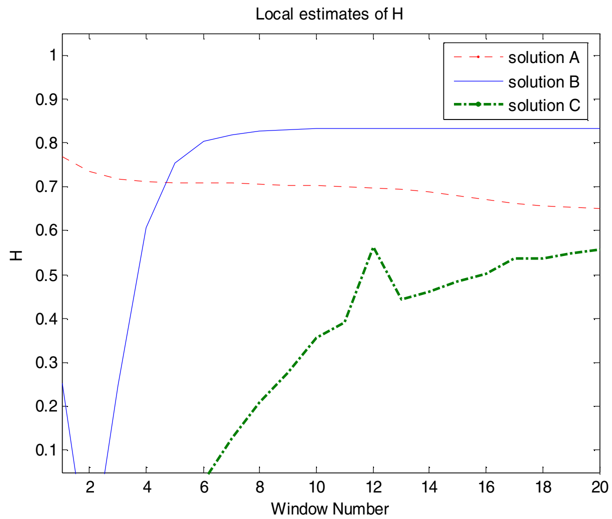


Fig. 6 Comparison of local Hurst parameter estimation using FrFT. Applied to biocorrosion potential noise of stainless steel electrode in 3 different artificial saliva for 24 hours.

The results are summarized in Fig. 6, where the local estimates of Hurst parameters of three different solutions are clearly distinguishable. One can easily tell the difference between the three different solutions. Therefore, the local estimation is suitable for characterizing the different self-similar long time series, while the global one fails.

VI. CONCLUSION

In this paper, we proposed a local analysis method using fractional Fourier transform. We used biocorrosion electrochemical noise data to illustrate how the FrFT based estimation method outperforms the other methods including both local and global schemes. We have shown that the fractional Fourier transform is very useful in studying long range dependence. The Husrt parameter estimation method based on FrFT spectrum is robust and reliable. It is more accurate to analyze the long range dependence locally in time, and the FrFT based method can better estimate the Hurst parameter of very long time series. In particular, local estimation provides a richer picture of the data. Our results also show that the FrFT based method is superior to the wavelet based method as far as the non-stationarity is concerned.

Our experiments show that biocorrosion data may possess long-range dependence on a certain range of time scale. Globally estimating the long biocorrosion data may result in misleading Hurst parameter. Choosing the right scales may help characterize the long range dependence of the biocorrosion data.

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REFERENCES

[1] YangQuan Chen, "Fractional order signal processing in biology/biomedical signal analysis", *Fractional Order Calculus Day at Utah State University*, April 19, 2005. http://mechatronics.ece.usu.edu/foc/event/FOC_Day@USU/FOSP-bioengineering.pdf

[2] S. Stoev, M. Taqqu, C. Park, and J. S. Marron, "On the wavelet spectrum diagnostic for Hurst parameter estimation in the analysis of internet traffic," *Computer Networks*, 2004, submitted for publication.

[3] S. Stoev, M. Taqqu, C. Park, G. Michailidis and J. S. Marron, "LASS: a tool for the local analysis of self-similarity in internet traffic," *SAMSI Tech. Rep.* No. 2004-7.

[4] P.M. Robinson, "Gaussian semiparametric estimation of long range dependence," *Annals of Statistics*, 1995, 23:1630-1661.

[5] Rongtao Sun, Nikita Zaveri, Yangquan Chen, Anhong Zhou, Nephi Zufelt, "Electrochemical noise signal processing using R/S analysis and fractional Fourier transform," *FDA 2006*.

[6] J. Geweke and S. Porter-Hudak. The estimation and application of long memory time series models. *J. Time Ser. Anal.*, 4:221–238, 1983.

[7] S. Stoev, M. Taqqu, C. Park, and J. Marron, "Strenghts and limitations of the wavelet spectrum method in the analysis of internet traffic," *Tech. Rep.* 2004-8, SAMSI, Mar. 2004. Research Triangle Park, NC, USA.

[8] H. M. Ozaktas and D. Mendlovi, "Fractional Fourier transformations and their optical implementation: Part II," ***J.** Opt. Soc. Amer. A,* vol. 10, pp. 2522-2531, 1993.

[9] Rongtao Sun, Yangquan Chen, Nikita Zaveri, Anhong Zhou, "Fractional Fourier transform based analysis of long range dependence on bioelectrochemical corrosion signals," submitted to *IEEE Signal Processing Letters*, 2006.

[10] D. Veitch and P. Abry, "A wavelet based joint estimator of the parameters of long-range dependence," *IEEE Trans. Info. Theory special issue on "Multiscale Statistical Signal Analysis and its Applications",* vol. 45, no. 3, April 1999.

[11] Nikita. Zaveri, Rongtao Sun, Yun Peng, Anhong Zhou, YangQuan Chen, "Biocorrosion study of stainless steel using electrochemical, AFM, and SEM," *11th Annual Institute of Biological Engineering Meeting*, Mar. 2006.

[12] Nikita. Zaveri, Rongtao Sun, Anhong Zhou, YangQuan Chen, Y. Peng, Nephi Z, "Evaluation of microbial influenced corrosion with the help of electrochemical noise analysis," to be submitted to *Electrochemical Acta,*, 2006.

[13] B. Racheva-Iotova and G. Samorodnitsky, "Long range dependence in heavy tailed stochastic processes," in *Handbook of Heavy Tailed Distributions in Finance*, Elsevier: Amsterdam, 2003.

[14] H. M. Ozaktas, M. A. Kutay, Z. Zalevsky. *The Fractional Fourier transform with Applications in Optics and Signal Processing*. John Wiley & Sons, 2000.

[15] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, "Convolution, filtering, and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms," *J. Opt. Soc. Amer. A,* vol. 11, pp. 547-559, 1994.

[16] Y. Li, H. H. Szu, Y. Sheng, H. J. Caulfield, "Wavelet processing and optics," *Proc. SPIE,* vol. 2051, pp. 130-141, 1993.

[17] P. Abry, and D. Veitch, "Wavelet analysis of long range dependent traffic,' *IEEE Trans. Info. Theory*, 44: 2—15, 1998.

[18] Bill Davidson, *Hurst exponent,* Matlab Central File Exchange, Feb. 2006. Avaliable online: http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=9842&objectType=file