

# Robust controllability of interval fractional order linear time invariant systems<sup>☆</sup>

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## Abstract

We consider uncertain fractional-order linear time invariant (FO-LTI) systems with interval coefficients. Our focus is on the robust controllability issue for interval FO-LTI systems in state-space form. We revisit the controllability problem for the case when there is no interval uncertainty. It turns out that the controllability check for FO-LTI systems amounts to checking the controllability of conventional integer order state space. Based on this fact, we further show that, for interval FO-LTI systems, the key is to check the linear dependency of a set of interval vectors. Illustrative examples are presented. © 2006 Elsevier B.V. All rights reserved.

*Keywords:* Fractional order systems; Robust controllability; Interval linear time invariant systems; Interval matrix; Linear dependency of interval vectors

## 1. Introduction

Based on fractional order calculus [1–4], fractional order dynamic systems and controls have been gaining increasing attention in research communities [5–9]. Pioneering works in applying fractional calculus in dynamic systems and controls include [10–13] while some recent developments can be found in [14–16].

Stability and controllability concepts are fundamental to any dynamic control systems including fractional order control systems [17,18]. In [19–24], stability results of fractional order control systems were presented while in [25], the first discussion about the controllability of fractional order control systems can be found. For interval FO-LTI systems, the first result on stability was discussed in [26] and further in [27] with even interval uncertainties (in the fractional order!). However, the controllability issue for interval FO-LTI systems has never been addressed. In this paper, we will present a method for checking the robust controllability for FO-LTI systems in the state space form. Based on the results of [28,29], we address the robust controllability issue via a sufficient linear independency condition of interval vectors. Note that, nobody has presented

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any property about the interval vectors except [28] although the interval vector concepts have been introduced in [30,31]. Furthermore, in robust control, the model uncertainty has been effectively and popularly handled by “interval” concept. Great amount of literatures are available under the term “interval” such as interval algebra [30,31], interval polynomial [32,33], Schur stability of interval matrices [34,35], Hurwitz stability of interval matrices [36–38], interval polynomial matrices [39], eigenvalues of interval matrices [40–42], and robust control with parameter uncertainty [43,44]. It is obviously beneficial to consider interval fractional order system as in [27,26]. For the ease of our presentation, we first re-visit the controllability issue of FO-LTI mainly based on [25]. Then, we briefly present the robust controllability issue of interval FO-LTI systems based on the concept of linear dependency of inter vectors [28]. Some examples will be given for illustrations.

## 2. Controllability of FO-LTI systems revisited

We adopt the Caputo definition for fractional derivative of order  $\alpha$  of any function  $f(t)$  [45,46]:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha+1-n}} d\tau, \quad (n - 1 < \alpha \leq n). \quad (1)$$

Based on the definition of (1), the Laplace transform of the fractional derivative is

$$\mathcal{L}\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}. \quad (2)$$

In general, an LTI FOS can be described by the differential equation or the corresponding transfer function of non-commensurate real orders of the following form:

$$G(s) = \frac{b_m s^{\beta_m} + \dots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \dots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \quad (3)$$

where  $\alpha_k, \beta_k$  ( $k = 0, 1, 2, \dots$ ) are real numbers and without loss of generality they can be arranged as  $\alpha_n > \dots > \alpha_1 > \alpha_0, \beta_m > \dots > \beta_1 > \beta_0$ .

In the particular case of *commensurate order* systems, it holds that,  $\alpha_k = \alpha k, \beta_k = \alpha k, (0 < \alpha < 1), \forall k \in \mathbb{Z}$ , and the transfer function has the following form:

$$G(s) = K_0 \frac{\sum_{k=0}^M b_k (s^\alpha)^k}{\sum_{k=0}^N a_k (s^\alpha)^k} = K_0 \frac{Q(s^\alpha)}{P(s^\alpha)}. \quad (4)$$

With  $N > M$ , the function  $G(s)$  becomes a proper rational function in the complex variable  $s^\alpha$  which can be expanded in partial fractions of the following form:

$$G(s) = K_0 \left[ \sum_{i=1}^N \frac{A_i}{s^\alpha + \lambda_i} \right], \quad (5)$$

where  $\lambda_i$  ( $i = 1, 2, \dots, N$ ) are the roots of the polynomial  $P(s^\alpha)$  or the system poles which are assumed to be simple without loss of generality. Then, it is straightforward to consider the following fractional order LTI system in state-space form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad (6)$$

where  $\alpha \in (0, 1], x \in \mathbb{R}^n, u \in \mathbb{R}^r, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, \text{rank}(B) = r$ .

Similar to the conventional controllability concept [17], the controllability of (6) is defined as follows:

**Definition 2.1.** The FO-LTI system (6) is said to be controllable on  $[t_0, t_f]$  iff for any initial state  $x(t_0)$  and final state  $x(t_f)$ , there exists a control function  $u(t)$  defined on  $[t_0, t_f]$  which can drive the initial state  $x(t_0)$  to the final state  $x(t_f)$ .

In what follows, we will show that, the controllability condition is the same as the integer order case. First, the solution of (6) is given by

$$X(s) = (s^\alpha I - A)^{-1} s^{\alpha-1} x(t_0) + (s^\alpha I - A)^{-1} B U(s) \quad (7)$$

in Laplace  $s$ -domain and

$$x(t) = E_{\alpha,1}(At^\alpha)x(t_0) + \int_{t_0}^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(A(t - \tau)^\alpha)Bu(\tau) d\tau \quad (8)$$

in time-domain where  $E_{\alpha,\beta}(z)$  is the Mittag-Leffler function in two parameters defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \beta > 0), \quad (9)$$

a generalization of exponential function, i.e.,

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Now, for any given  $t_0, t_f$  and the states  $x(t_0)$  and  $x(t_f)$ , let us see under what condition there exists a unique control function  $u(t)$  for  $t \in [t_0, t_f]$ . From (8),

we have

$$x(t_f) - E_{\alpha,1}(At_f^\alpha)x(t_0) = \int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} E_{\alpha,\alpha}(A(t_f - \tau)^\alpha)Bu(\tau) d\tau. \quad (10)$$

Note that

$$E_{\alpha,\alpha}(At^\alpha) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha k + \alpha)} = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + \alpha)} A^k.$$

With Cayley–Hamilton theorem,  $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$  can be written in the following form:

$$t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + \alpha)} A^k = \sum_{k=0}^{n-1} c_k(t)A^k. \quad (11)$$

So, (10) can be written as

$$x(t_f) - E_{\alpha,1}(At_f^\alpha)x(t_0) = \sum_{k=0}^{n-1} A^k B \int_{t_0}^{t_f} c_k(t_f - \tau)u(\tau) d\tau. \quad (12)$$

In matrix form, the above equation becomes

$$x(t_f) - E_{\alpha,1}(At_f^\alpha)x(t_0) = [B|AB|A^2B|\dots|A^{n-1}B] \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix}, \quad (13)$$

where  $d_k = \int_{t_0}^{t_f} c_k(t_f - \tau)u(\tau) d\tau$ . Note that, since  $t_0$ ,  $t_f$  and the states  $x(t_0)$  and  $x(t_f)$  are arbitrary, to have a unique solution of  $u(t)$ , the necessary and sufficient condition is clearly that the so-called controllability matrix  $\mathcal{C} = [B|AB|A^2B|\dots|A^{n-1}B]$  has a full rank, which is the same condition as in the conventional integer order case [17].

### 3. Robust controllability test of FO-LTI systems using interval vector concept

The robust controllability problem of uncertain integer-order linear system has been studied in [47,48] with significant amount of conservatism. In [29], an alternative method was developed based on interval vectors [28], which is very simple but much less conservative. In this paper, we will follow [29] to

present a robust controllability test procedure for FO-LTI systems using the interval vector concept.

The following FO-LTI uncertain system is considered:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad (14)$$

which is similar to (6). However,  $A \in A^I = [\underline{A}, \overline{A}]$  and  $B \in B^I = [\underline{B}, \overline{B}]$ . We call the interval uncertain fractional order system (14) controllable if  $\text{rank}(C^I) = n$ , where

$$C^I = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \dots, \underbrace{A^I \otimes \dots \otimes A^I}_{n-r} \otimes B^I],$$

which is  $n \times (n - r + 1) \cdot r$  interval matrix. Note symbol  $\otimes$  stands for the multiplication of intervals as explained in what follows. For convenience,  $m \equiv (n - r + 1) \cdot r$ . In fact, the main source of conservatism of [47,48] is due to the fact that they used  $C^I$  without any modification for the controllability test. We can avoid this problem by using interval vector concept.

For the sake of the interval approach, some basic notations and definitions are given as follows.

**Definition 3.1.** A real interval scalar  $x^I$  is defined as  $x^I := [\underline{x}, \overline{x}]$ , where  $\underline{x}, \overline{x} \in \mathcal{R}$  and for all  $x \in x^I$ , there exists a corresponding  $\lambda$  such that  $x = \lambda \overline{x} + (1 - \lambda) \underline{x}$  with  $0 \leq \lambda \leq 1$ ,  $\lambda \in \mathcal{R}$ . The  $n$ -dimensional real column interval vector  $\mathbf{x}^I$  is defined as  $\mathbf{x}^I := (x_1^I, \dots, x_n^I)^T$  and the  $n \times m$  dimensional real interval matrix is defined from the interval vectors as

$$X^I := (\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_m^I).$$

The interval vector and interval matrix can be written as  $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$  and  $X^I = [\underline{X}, \overline{X}]$ . Or, they can be written as:  $\mathbf{x}^I = [\mathbf{x}_0 - \Delta \mathbf{x}, \mathbf{x}_0 + \Delta \mathbf{x}]$  and  $X^I = [X_0 - \Delta X, X_0 + \Delta X]$ , where  $\mathbf{x}_0 = (\overline{\mathbf{x}} + \underline{\mathbf{x}})/2$ ,  $X_0 = (\overline{X} + \underline{X})/2$ ,  $\Delta \mathbf{x} = (\overline{\mathbf{x}} - \underline{\mathbf{x}})/2$ , and  $\Delta X = (\overline{X} - \underline{X})/2$ .

Based on [30,31], the following interval arithmetics are used.

**Definition 3.2.** The intersection of two real interval scalars  $x^I$  and  $y^I$  is defined as  $x^I \cap y^I := \{z|z \in x^I \text{ and } z \in y^I\}$ . The union of two real interval scalars  $x^I$  and  $y^I$  is defined as  $x^I \cup y^I := \{z|z \in x^I \text{ or } z \in y^I\}$ .

**Definition 3.3.** The addition of two real interval scalars  $x^I$  and  $y^I$  is defined and calculated as  $x^I \oplus y^I = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ , the subtraction is  $x^I \ominus y^I =$

$[\underline{x} - \bar{y}, \bar{x} - \underline{y}]$ , and the multiplication is

$$x^I \otimes y^I = [\min\{\underline{x}y, \underline{x}\bar{y}, \bar{x}y, \bar{x}\bar{y}\}, \max\{\underline{x}y, \underline{x}\bar{y}, \bar{x}y, \bar{x}\bar{y}\}].$$

The division should be carefully defined as [31]

$$\begin{aligned} \frac{1}{x^I} &= \emptyset \quad \text{if } x^I = [0, 0] \\ &= \infty \quad \text{if } x^I = (0, 0^+] \\ &= -\infty \quad \text{if } x^I = [0^-, 0) \\ &= \left[ \frac{1}{\bar{x}}, \frac{1}{\underline{x}} \right] \quad \text{if } x^I > 0 \\ &= \left[ \frac{1}{\underline{x}}, \frac{1}{\bar{x}} \right] \quad \text{if } x^I < 0 \\ &= [-\infty, \infty] \quad \text{if } \underline{x} < 0 \text{ and } \bar{x} > 0. \end{aligned} \quad (15)$$

Then, the division of two interval scalars is simply defined and calculated as  $x^I \oslash y^I = x^I \otimes 1/y^I$ .

The interval arithmetics of a real interval scalar by itself should be distinguished from the arithmetics of two different scalar intervals. For the LTI system,<sup>1</sup> we use the following definitions:

**Definition 3.4.** If  $x^I$  is not time dependent (i.e., time invariant), the addition of a real interval scalar  $x^I$  is defined and calculated as  $x^I \oplus x^I = [\underline{x} + \underline{x}, \bar{x} + \bar{x}]$ , the subtraction is  $x^I \ominus x^I = 0$ , and the multiplication is  $x^I \otimes x^I = [\alpha^2, \beta^2]$ , where  $\alpha = \min\{|\underline{x}|, |\bar{x}|\}$ ;  $\beta = \max\{|\underline{x}|, |\bar{x}|\}$ . The division is defined as  $x^I \oslash x^I = 1$  if  $x^I \neq [0, 0]$ .

In linear algebra, the following linear (in)dependency condition of the linear vectors is popularly used.

**Definition 3.5.** Without interval, when  $n$  different vectors are given as  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , they are called in linearly independent iff there exist only trivial solutions ( $a_1 = a_2 = \dots = a_n = 0$ ) such that  $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n = 0$ . Otherwise, they are linearly dependent. If they are linearly independent, any  $\mathbf{x}_i$  cannot be produced by any combinations of other vectors.

Now, with the basic definitions given above, we define the linear (in)dependency of interval vectors.

**Definition 3.6.** With interval, let us suppose we have  $n$  different interval column vectors given as  $\mathbf{x}_1^I, \dots, \mathbf{x}_n^I$ . They are called in linearly independent iff there exist only trivial solutions ( $a_1 = a_2 = \dots = a_n = 0$ ) such that  $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n = 0$  for all  $\mathbf{x}_i \in \mathbf{x}_i^I$ ,  $i = 1, \dots, n$ . Other-

Table 1  
Algorithm for linear (in)dependency test of interval vectors

- 
- (i) Make an interval matrix using interval vectors like (17).
  - (ii) Find sub square matrices as defined in Definition 5.1.
  - (iii) In each sub-matrix, find center matrix and radius matrix like (19) and (20).
  - (iv) For linear independency test, use Theorem 5.1 or Corollary 3.
  - (v) For linear dependency test, use Corollary 5.
- 

wise, we say that the interval vectors are in linearly dependent.

Now, based on above definitions, an algorithm for checking the linear dependency and independency of interval vector is provided in Table 1, which is a brief summary of the results of [28] (see also appendix).

First let us consider the case without interval

$$\frac{d^2 x(t)}{dt^2} = A_0 x + B_0 u \quad (16)$$

and the corresponding controllability matrix is

$$C_0 = [B_0, A_0 B_0, (A_0)^2 B_0, \dots, (A_0)^{n-r} B_0].$$

If the system is controllable,  $\text{rank}(C_0) = n$ . To distinguish the interval case from the non-interval case, let us suppose that the rank of following sub-matrix of  $C_0$

$$C'_0 = [B_0, A_0 B_0, (A_0)^2 B_0, \dots, (A_0)^{n-r-q} B_0],$$

where  $q \geq 1$ , is  $n$  (i.e.,  $\text{rank}(C'_0) = n$ ). Then, with no interval, it is always true that  $\text{rank}(C'_0) = \text{rank}(C_0) = n$ . Now, let us include interval. In this case, we have to check the rank of  $C^I$ , but since  $C^I$  is a  $n \times m$  interval matrix, it is not easy to find the rank of  $C^I$ . Thus, in [47,48], inevitably, they tried to find some inequality conditions in matrix norm to guarantee the sufficient conditions of LTI interval system (see Eq. (3.9) in [48] and Eq. (10) in [48]). Using these inequalities, they found the upper boundaries for sufficient condition, but in this upper boundary calculation, the formula is very conservative (see the derivation of Theorem 1 of [47] and Eq. (3.6) of [48]). So, even there is ignorable interval uncertainty in  $(C^I)'$ , which is defined as

$$(C^I)' = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \dots, \underbrace{A^I \otimes \dots \otimes A^I}_{n-r-q} \otimes B^I],$$

the overall upper bounds are calculated from the maximum interval uncertainty of  $C^I$ . So, the controllability checking methods of [47,48] instinctively

<sup>1</sup>For linear time varying case, we have to use Definition 3.3.

are conservative since their approach must investigate the sufficient conditions based on  $C^I$  using maximum interval uncertain element of the interval  $A^I$ . This is the main reason why their methods are so conservative.

However, if we can check the rank of  $C^I$  directly, the result will be much less conservative, which can be done by checking the linear independency property of the interval vectors. Based on the algorithm given in Table 1, which is the summary of appendix, easily we can check the robust controllability. For convenience, the controllability checking methods of the uncertain LTI system are formulated in the following theorem:

**Theorem 3.1.** *If the controllability matrix  $C^I$  satisfies the linear independency conditions of Theorem 5.1, then the uncertain interval system is controllable.*

**Proof.** Since the interval system is controllable if its controllability matrix has rank  $n$  and the full rank condition is equivalent to the linear independency condition, the proof is immediate.  $\square$

**Corollary 1.** *If the controllability matrix  $C^I$  satisfies the linear independency conditions of Corollary 3, then the uncertain interval system is controllable.*

**4. Illustrative examples**

In this section, we compare the test results using the same examples given in [47].

**4.1. Example 1**

$$A \in A^I = \begin{pmatrix} 1 \pm 0.05 & 0 & 0 \\ 0 & 1 \pm 0.04 & 1 \pm 0.03 \\ 0 & -2 \pm 0.08 & 4 \pm 0.4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The controllability matrix  $C$  is calculated from the interval arithmetics as

$$C \in C^I = \begin{pmatrix} 1 & 0 & 1 \pm 0.05 & 0 \\ 0 & 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 & 4 \pm 0.4 \end{pmatrix}.$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 & 0 & 1 \pm 0.05 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad S^2 \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 4 \pm 0.4 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 & 1 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 0 & 4 \pm 0.4 \end{pmatrix};$$

$$S^4 \in \begin{pmatrix} 0 & 1 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 1 & 0 & 4 \pm 0.4 \end{pmatrix}.$$

Then, from  $S^2$ , we have

$$S_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}; \quad \Delta S^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.03 \\ 0 & 0 & 0.4 \end{pmatrix}.$$

Therefore, since  $S_0^2$  is nonsingular and  $\rho(\|(S_0^2)^{-1}\|\Delta S^2) = 0.03 < 1$ , easily we confirm that the interval system is controllable. However, in [47], they conclude that their method cannot check the controllability directly, which is due to the conservatism of their method. Clearly, our method is much less conservative. In [47], the following sign variant problem was given.

**4.2. Example 2**

$$A \in A^I = \begin{pmatrix} 0 \pm 0.05 & 0 & 0 \\ 0 & 1 \pm 0.04 & 1 \pm 0.03 \\ 0 & 0 \pm 0.08 & 0 \pm 0.4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The controllability matrix  $C$  is calculated as

$$C \in C^I = \begin{pmatrix} 1 & 0 & 0 \pm 0.05 & 0 \\ 0 & 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 & 0 \pm 0.4 \end{pmatrix}.$$

So, we have four sub-square matrices

$$S^1 \in \begin{pmatrix} 1 & 0 & 0 \pm 0.05 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad S^2 \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 1 & 0 \pm 0.4 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 & 0 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 0 & 0 & 0 \pm 0.4 \end{pmatrix};$$

$$S^4 \in \begin{pmatrix} 0 & 0 \pm 0.05 & 0 \\ 0 & 0 & 1 \pm 0.03 \\ 1 & 0 & 0 \pm 0.4 \end{pmatrix}.$$

From  $S^2$ ,  $S_0^2$  is nonsingular and  $\rho(\|(S_0^2)^{-1}\|\Delta S^2) = 0.03 < 1$ . So, regardless the sign variation, easily we find that the interval system is controllable. However, in [47], they used controller  $K$  to guarantee the controllability, but as resulted from our method, the system is already controllable. So, their approach requires the extra work, which is not necessary in our method. The following example includes the interval in  $B$ .

4.3. Example 3

$$A \in A^I = \begin{pmatrix} 1 \pm 0.02 & 0 & 0 \\ 0 & 1 \pm 0.02 & 1 \pm 0.02 \\ 0 & -2 \pm 0.05 & 4 \pm 0.09 \end{pmatrix},$$

$$B \in B^I = \begin{pmatrix} 1 \pm 0.025 & 0 \\ 0 & 0 \\ 0 & 1 \pm 0.02 \end{pmatrix}.$$

The controllability matrix  $C$  is calculated as

$$C \in C^I = \begin{pmatrix} 1 \pm 0.025 & 0 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 0 & 1 \pm 0.0404 \\ 0 & 1 \pm 0.02 & 0 & 4 \pm 0.1718 \end{pmatrix}.$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 \pm 0.025 & 0 & 1 \pm 0.0455 \\ 0 & 0 & 0 \\ 0 & 1 \pm 0.02 & 0 \end{pmatrix};$$

$$S^2 \in \begin{pmatrix} 1 \pm 0.025 & 0 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 0 & 1 \pm 0.02 & 4 \pm 0.1718 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 \pm 0.025 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 0 & 0 & 4 \pm 0.1718 \end{pmatrix};$$

$$S^4 \in \begin{pmatrix} 0 & 1 \pm 0.0455 & 0 \\ 0 & 0 & 1 \pm 0.0404 \\ 1 \pm 0.02 & 0 & 4 \pm 0.1718 \end{pmatrix}.$$

Since from  $S^2$ ,  $S_0^2$  is nonsingular and  $\rho(\|(S_0^2)^{-1}\|\Delta S^2) = 0.04 < 1$ , the system is controllable. From these examples, it is clear our method is much simple and much less conservative than the existing method in checking the robust controllability of the uncertain LTI system. The robust observability is dual to the robust controllability problem and can be verified based on our method easily.

5. Conclusion

In this paper, we have considered the robust controllability problem of uncertain fractional-order linear time invariant (FO-LTI) systems with interval coefficients. We first re-visited the controllability problem for the case when there is no interval uncertainty. It turns out that the stability check for FO-LTI systems amounts to checking the conventional integer order state space using the same state matrix  $A$  and the input coupling matrix  $B$ . Based on this fact, we further show that, for interval FO-LTI systems, the key is to check the linear dependency of a set of interval vectors. Illustrative examples are presented.

Appendix: Linear dependency and independency of interval vectors

A simple but very effective sufficient condition for checking the linear (in)dependency of interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$ , where an  $\mathbf{x}_i^I$  is an interval vector in  $\mathcal{R}^m$ , was developed in [28]. For convenience, the following  $m \times n, m > n$  matrix is considered:

$$M = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ m_{31} & m_{32} & \dots & m_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{pmatrix}. \tag{17}$$

Then, we check all possible  $n \times n$  square sub-matrices, which are defined as follows:

**Definition 5.1.** A set of all sub-matrices  $S_M = \{S^i, i = 1, \dots, k\}$  is called square set and each sub matrix  $S^i$  is called sub-square matrix.

Now, we consider the interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$ . Let us write these interval vectors in an interval matrix form such as:

$$X^I = (\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I). \tag{18}$$

Then,  $X^I$  is an  $m \times n$  interval matrix, so based on Definition 5.1, the corresponding square set of  $X^I$  can be found as:  $S_X = \{S^i, i = 1, \dots, k\}$  where

$$k = \binom{m}{n} = \frac{m(m-1)(m-2) \cdots (m-n+1)}{n!}.$$

Here, we further define the center square matrices  $S_{X_c}$  and calculate them as

$$S_{X_c} = \left\{ S_0^i = \frac{S^i + \bar{S}^i}{2}, i = 1, \dots, k \right\}, \tag{19}$$

and define the radius square matrices  $\Delta S_X$  and calculate them as

$$\Delta S_X = \left\{ \Delta S^i = \frac{\bar{S}^i - S^i}{2}, i = 1, \dots, k \right\}. \tag{20}$$

For our main result, notating the absolute value of a matrix  $A$  by  $\|A\| = (|a_{ij}|)$ , the following lemma can be adopted from [49].

**Lemma 2.** For interval matrix  $X^I$ , let its center matrix  $X_0$  be nonsingular and the spectral radius  $\rho(\|(X_0)^{-1}\|\Delta X) < 1$ , then  $X^I$  is nonsingular.

Now, for the linear independency test of the interval vector set, we suggest the following theorem:

**Theorem 5.1.** For  $S^I \in S_X$ , if there exists at least one corresponding  $S_0 \in S_{X_c}$  and  $\Delta S \in \Delta S_X$  such that  $S_0$  is nonsingular and  $\rho(\|(S_0)^{-1}\|\Delta S) < 1$ , then the interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$  are linearly independent.

**Proof.** Let us consider  $X^I = (\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I)$ , which is an  $m \times n, m > n$ , interval matrices composed of the interval vectors. It is a fact that the column vectors are linearly independent if (and only if in the point of “rank”) the rank of  $X^I$  is  $n$ . Also from the fact that the row rank is equal to the column rank, so if  $S^I$  has rank  $n$ , then the column rank of  $X^I$  is also  $n$ . Therefore, if any one of  $S^I \in S_X$  has row rank  $n$ , then  $X^I$  has  $n$  column rank. So, by Lemma 2, for  $S_0$  and  $\Delta S$  corresponding to  $S^I$ , if  $S_0$  is nonsingular and  $\rho(\|(S_0)^{-1}\|\Delta S) < 1$ , then  $X^I$  has full column rank, because the nonsingular condition is equivalent to the full rank condition. Thus, since the full

column rank indicates the linear independency, the proof is completed.  $\square$

However, although Theorem 5.1 is represented in a simple form, the result could be conservative in checking the condition  $\rho(\|(S_0)^{-1}\|\Delta S) < 1$ , because  $\|(S_0)^{-1}\|$  is used. To reduce the conservatism, the following result can be obtained based on Theorem 5.1.

**Corollary 3.** For at least one  $S^I \in S_X$  and for its corresponding  $S_0 \in S_{X_c}$  and  $\Delta S \in \Delta S_X$ , if there exists a matrix  $R$  such that

$$\rho(\|I - RS_0\| + \|R\|\Delta S) < 1,$$

then the interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$  are linearly independent.

**Proof.** The proof can be completed by the proof of Theorem 5.1 and Theorem 3.1 of [49].  $\square$

Using the proof of Theorem 5.1 and using the results of [49], we also can find the sufficient condition for linear dependency of the interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$ . Let us use the following lemma for this purpose.

**Lemma 4.** For interval matrix  $X^I$ , there exist a matrix  $R$  and a natural number  $p$  such that

$$(I + \|I - X_0R\|)_p \leq (\Delta X\|R\|)_p,$$

where  $p \in \{1, \dots, n\}$  and  $(\cdot)_p$  represents  $p$ th column, then interval matrix  $X^I$  is singular.

**Proof.** See theorem 3.3 of [49].  $\square$

**Corollary 5.** For all  $S^I \in S_X$  and for all its corresponding  $S_0 \in S_{X_c}$  and  $\Delta S \in \Delta S_X$ , if there exist a matrix  $R$  and a natural number  $p$  such that

$$(I + \|I - S_0R\|)_p \leq (\Delta S\|R\|)_p,$$

then the interval vectors  $\mathbf{x}_1^I, \mathbf{x}_2^I, \dots, \mathbf{x}_n^I$  are linearly dependent.

**Proof.** Theorem 5.1 shows that the interval vectors are linearly independent if there exists at least one  $S^I$  such that the conditions of Theorem 5.1 hold. So, to eliminate the case of Theorem 5.1, we have to check all  $S^I \in S_X$  for the linearly dependent test. Hence, by checking all  $S^I$  and based on the proof of Theorem 5.1 and Lemma 4, the proof of Corollary 5 can be completed.  $\square$

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