Exact maximum singular value calculation of an interval matrix

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***Abstract*— In this note, we present a method for calculating the maximum singular value of an interval matrix. First, we provide an algorithm for calculating the maximum singular value of a square interval matrix. Then, based on this algorithm, we extend the result to non-square interval matrix case and to the case of computing the minimum singular value. Through numerical examples, the validity of the suggested methods is illustrated. Particularly, we compare the newly-proposed method with an existing method to show that the new method finds the correct bound of the maximum singular value with no exception.**

***Index Terms*— Maximum singular value, interval matrix.**

I. INTRODUCTION

A great amount of literature is available for interval uncertain matrix and its stability conditions [1], [2], [3], [4]. In particular, the Hurwitz stability [5], [6], the Schur stability [7], [8], and the eigenvalue boundary problem with perturbation [3], [4], [9], [10] have been well studied and formulated. However, some fundamental interval computational problems such as "power of an interval matrix," "analytical stability condition of an interval polynomial matrix," and "maximum singular value bound of an interval matrix" have not been well solved yet. For more detailed discussions about these problems and for some initial results, refer to [11], [12]. This note provides a solution for determining the bound of the maximum singular value of an interval matrix for the first time based on authors' best knowledge.

From literature search, in fact, boundaries of singular values of an interval matrix were studied in [13]. However, the sign of eigenvectors was limited to be unchanged with interval perturbation. Thus, the algorithm presented in [13] was developed based on some restrictive assumptions. In H_∞ robust control, the singular value of an uncertain plant is popularly used under the name of structured singular value (SSV) [14] for designing a robust controller. However, in this traditional robust control, an inequality of the maximum singular value is estimated under the condition that the uncertainty is structured. In this note, we provide a generalized method, which can be used for finding the maximum singular value of a general (unstructured) interval matrix. As possible applications of the results, the monotonic convergence condition of an uncertain discrete-time linear system can be effectively checked[1] and possibly it can be used for μ synthesis [14] of a robust control system.

This paper is structured as follows. In Section II, we provide the main result for square matrices. Then, in Section III we extend the result to non-square matrices, and in Section IV we consider the minimum singular value of an interval matrix. In Section V, numerical examples are offered for demonstration purpose and conclusion is given in Section VI.

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[1]For instance, in iterative learning control, the monotonic convergence of an uncertain discrete system is checked by the maximum singular value (2-norm) of an interval impulse-response matrix [15].

II. MAIN RESULTS

For our main results, we make use of Hertz's idea for finding extreme eigenvalues of a *symmetric* interval matrix [2]. In this paper, let us consider a real square non-symmetric interval matrix such as:

$$A^I = [a^I_{ij}], \quad a^I_{ij} := [\underline{a_{ij}}, \overline{a_{ij}}], i, j = 1, \dots, n \qquad (1)$$

where a^I_{ij} is an element of interval matrix A^I, $\underline{a_{ij}}$ is the lower boundary of an interval a^I_{ij}, and $\overline{a_{ij}}$ is the upper boundary of an interval a^I_{ij}. If we define the lower boundary matrix and the upper boundary matrix as $\underline{A} = [\underline{a_{ij}}]$ and $\overline{A} = [\overline{a_{ij}}]$ respectively, the interval matrix can then be written as $A^I := [A^o - \Delta, A^o + \Delta]$, where the center matrix (A^o) and the radius matrix (Δ) are defined as

$$A^o = \frac{1}{2}(\overline{A} + \underline{A}); \ \Delta = \frac{1}{2}(\overline{A} - \underline{A}).$$

In fact, the upper boundary of singular values of an interval matrix can be estimated using the relationship $\sigma_i(A^I) = \sqrt{\lambda_i((A^I)^T \otimes A^I)}$ where $\otimes$ represents multiplication of interval matrices, σ_i is the singular value, λ_i is the eigenvalue, and $i = 1, 2, \dots, \mathrm{rank}(A)$. However, as commented in [13], the results of this approach will be quite conservative. In this paper, we suggest using the following relationship between singular values and eigenvalues:

$$\begin{aligned} \sigma_i(A) &= \text{Positive}\left(\lambda_i \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}\right) \\ &= \text{Positive}\left(\lambda_i(H)\right), \ A \in A^I \qquad (2) \end{aligned}$$

where $\text{Positive}(\cdot)$ considers only the positive part of $(\cdot)$. Obviously, H is a symmetric matrix and it is a member of the symmetric interval matrix

$$H^I = \begin{bmatrix} 0 & (A^I)^T \\ A^I & 0 \end{bmatrix}.$$

Hence, if we make use of the results of [2], there will be a way to find maximum singular value (denoted as $\overline{\sigma}$) of A^I. In the sequel, we briefly summarize our main idea and results. Based on [2], since H and H^I are symmetric matrices, we have the relationship:

$$x^T \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} x = x^T H x = \lambda^* \tag{3}$$

where x is the eigenvector corresponding to a particular eigenvalue λ^* and $x^T x = 1$. Let us divide x into two parts such as $x^T = [y^T, z^T]$. Then, $y_i = x_i,\ i = 1, \ldots, n$ and $z_i = x_{n+i},\ i = 1, \ldots, n$, and from (3), we obtain $\lambda^* = 2\left(\Sigma_{i=1}^n \Sigma_{j=1}^n a_{ji} y_i z_j\right)$. Therefore, the value of λ^* depends on signs of y_i and z_j. That is, the maximum of λ^* occurs at one of vertex points of a_{ij}, which is given as:

$$a_{ij} = \begin{cases} a_{ij} = \overline{a_{ij}} & \text{if} \quad y_i z_j \geq 0 \\ a_{ij} = \underline{a_{ij}} & \text{if} \quad y_i z_j < 0 \end{cases} \tag{4}$$

Now, since y and z are length-n vectors, we have a total number of 2^n different sign patterns for y and 2^n different sign patterns for z. For example, when $n = 3$, sign patterns of y and z will be $+++$, $++-$, $+-+$, $+--$, $-++$, $-+-$, $--+$, $---$. In this case, we have a total number of $2^3 \times 2^3 = 64$ combinations as shown in Table I and Table II.

TABLE I

32 SIGN PATTERNS WITH $\text{sign}(y_1) = +$ FOR 3×3 MATRIX.

y	z	y	z
$+++$	$+++$	$+-+$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$
$++-$	$+++$	$+--$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$

However, Table I and Table II produce the same vertex matrices set from A^I. Therefore, in our purpose, it will be enough to check a total number of 2^5 vertex matrices corresponding to Table I. These vertex matrices can be found easily. For example, in Table I, for sign pattern $+--$ of y and for the sign pattern $+-+$ of z, the sign of the corresponding-vertex matrix is defined by zy^T such as:

$$\begin{bmatrix} + \\ - \\ + \end{bmatrix} [+ \quad - \quad -] = \begin{bmatrix} + & - & - \\ - & + & + \\ + & - & - \end{bmatrix}. \tag{5}$$

Therefore, one of the corresponding-vertex matrices for the

TABLE II

32 SIGN PATTERNS WITH $\text{sign}(y_1) = -$ FOR 3×3 MATRIX.

y	z	y	z
$-++$	$+++$	$--+$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$
$-+-$	$+++$	$---$	$+++$
	$++-$		$++-$
	$+-+$		$+-+$
	$+--$		$+--$
	$-++$		$-++$
	$-+-$		$-+-$
	$--+$		$--+$
	$---$		$---$

calculation of maximum singular value is found as:

$$\begin{bmatrix} \overline{a_{ij}} & \underline{a_{ij}} & \underline{a_{ij}} \\ \underline{a_{ij}} & \overline{a_{ij}} & \overline{a_{ij}} \\ \overline{a_{ij}} & \underline{a_{ij}} & \underline{a_{ij}} \end{bmatrix}. \tag{6}$$

Based on the above discussion, for a general size n square interval matrix, we can develop an algorithm such as:

- **Step-1**: Produce the set of ± 1 vectors with $y_1 = 1$ of length n such as
$$Y = \{y \in R^n : y_1 = 1,\ |y_j| = 1,\ \text{for } j = 2, \ldots, n\}.$$
- **Step-2**: Produce the set of ± 1 vectors of length n such as
$$Z = \{z \in R^n :\ |z_j| = 1,\ \text{for } j = 1, \ldots, n\}.$$
- **Step-3**: Make $n \times n$ diagonal matrix T_y defined by $(T_y)_{ii} = y_i$ and $(T_y)_{ij} = 0$ for $i \neq j,\ i, j = 1, \ldots, n$ where $y \in Y$.
- **Step-4**: Make $n \times n$ diagonal matrix T_z defined by $(T_z)_{ii} = z_i$ and $(T_z)_{ij} = 0$ for $i \neq j,\ i, j = 1, \ldots, n$ where $z \in Z$.
- **Step-5**: Produce a matrix set $\mathcal{S}^v := \{A_{yz} : A_{yz} = A^o + T_y \Delta T_z\ \forall\ y \in Y \ \text{ and } \ \forall\ z \in Z\}$.
- **Step-6**: Find maximum singular values of all element of the finite set $\mathcal{S}^v$ and select the largest one as the maximum singular value of the interval matrix A^I.

Now, to summarize our idea in a compact form and to improve the clarity of the presentation, we make the following theorem.

Theorem 2.1: Given a square interval matrix A^I, the maximum singular value can be found from the reduced-vertex matrices set $\mathcal{S}^v$, i.e., $\max\{\overline{\sigma}(A),\ A \in A^I\} = \max\{\overline{\sigma}(S),\ S \in \mathcal{S}^v\}$.

Proof: Since H^I is a symmetric interval matrix, from

$\lambda^* = 2\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{n} a_{ji} y_i z_j\right)$ where $a_{ji} \in a_{ji}^I$, $\max\{\ a_{ji} y_i z_j\}$ occurs at one of vertex points of a_{ji}^I depending on sign of $y_i z_j$. Furthermore, since $\text{sign}(zy^T) = \text{sign}[(-z)(-y)^T]$ as illustrated from Table I and Table II, the proof is direct by the relationship (2). ■

III. Maximum singular value of non-square interval matrix

Results of the preceding section can be extended to the general non-square interval matrix case easily. Let us consider $m \times n$ interval matrix A^I. Then, H^I is $(m+n) \times (m+n)$ interval matrix. Now, introducing length-n vector y and length-m vector z, using the same procedure as in the square matrix case, we have $\lambda^* = 2\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{m} a_{ji} y_i, z_j\right)$. Then, we can find that there are a total number of 2^{m+n-1} possible combinations of vertex matrices that can be used for the calculation of the maximum singular value of a non-square interval matrix. For example, for 3×2 matrix, we have a total number of $2^3 \times 2^1$ combinations as shown in Table III.

TABLE III

16 sign patterns for 3×2 non-square matrix.

y	z	y	z
++	+++	+−	+++
	++−		++−
	+−+		+−+
	+−−		+−−
	−++		−++
	−+−		−+−
	−−+		−−+
	−−−		−−−

In Table III, for example, for the sign pattern $+-$ of y and for the sign pattern $+-+$ of z, the sign of vertex matrix is defined by zy^T such as:

$$\begin{bmatrix} + \\ - \\ + \end{bmatrix} \begin{bmatrix} + & - \end{bmatrix} = \begin{bmatrix} + & - \\ - & + \\ + & - \end{bmatrix}, \qquad (7)$$

which provides the corresponding-vertex matrix:

$$\begin{bmatrix} \overline{a_{ij}} & \underline{a_{ij}} \\ \underline{a_{ij}} & \overline{a_{ij}} \\ \overline{a_{ij}} & \underline{a_{ij}} \end{bmatrix}. \qquad (8)$$

A similar algorithm and a similar theorem as given in Section II can be developed for a non-square case. However, due to the simplicity, detailed discussions are omitted.

IV. Minimum singular value of an interval matrix

It is clear that the minimum singular value (denoted as $\underline{\sigma}$) of an interval matrix does not occur at one of the vertex matrices. For example, for the following interval matrix

$$A^I = \begin{bmatrix} [-3,3] & [-1,1] \\ [0,2] & [0,1] \end{bmatrix}, \qquad (9)$$

the minimum singular value is zero. However, if we use vertex matrices, the minimum singular value is bigger than zero. The following remark is provided for the calculation of the minimum singular value in some cases.

Remark 4.1: When an interval matrix is square and regular (i.e., nonsingular of an interval matrix [8]), the inverse of an interval matrix can be found using the method suggested in [16]. Then, from the relationship $\underline{\sigma}(A) = 1/\overline{\sigma}(A^{-1})$, we can find the minimum singular value of an interval matrix.

V. Illustrative Examples

A. Example-1: Non-square case

Let us test the non-square case first. For a non-square case, we use the following existing example from [13]:

$$A^I = \begin{bmatrix} [2,3] & [1,1] \\ [0,2] & [0,1] \\ [0,1] & [2,3] \end{bmatrix}.$$

Using the results given in Section III, we found the maximum singular value of A^I as 4.54306177572459, which is quite close to the value 4.543062 given in [13]. This result shows that our method can find the exact (without conservatism) upper boundary of the maximum singular value of an interval matrix. Note that the suggested scheme in this paper does not require any assumption for calculating the upper boundary of the maximum singular value of an interval matrix.

B. Example-2: Square case

Next, for the square matrix and to represent an exception of Deif's method [13], we test an interval matrix with the following center matrix and the radius matrix

$$A^o = \begin{bmatrix} -3.33 & -2.24 & 0.06 \\ 1.03 & -0.34 & 1.09 \\ -2.02 & -1.02 & 2.27 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 1.32 & 0.86 & 4.38 \\ 0.84 & 2.97 & 1.42 \\ 1.61 & 3.06 & 0.55 \end{bmatrix}.$$

Using the suggested method, we found that the maximum singular value of A^I is 9.8549, but from Deif's method, we have 9.7408. For demonstration purpose, we performed random tests. Fig. 1 shows the Monte-Carlo type random tests. In this figure, the dashed-dot line is the calculated maximum singular value from the suggested method (9.8549) and the solid line is the maximum singular value from Deif's method. Clearly there exist exceptions in the case of Deif's method, while the suggested method provides the exact boundary without an exception.

C. Example-3: Minimum singular value

For the minimum singular value, let us consider an interval matrix with the following center matrix and radius matrix

$$A^o = \begin{bmatrix} 1.5 & -0.01 & 3.4 \\ 7.1 & -3.4 & -1.3 \\ 2.1 & 0.01 & -7 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 0.75 & 0.005 & 1.7 \\ 3.55 & 1.7 & 0.65 \\ 1.05 & 0.005 & 3.5 \end{bmatrix}.$$

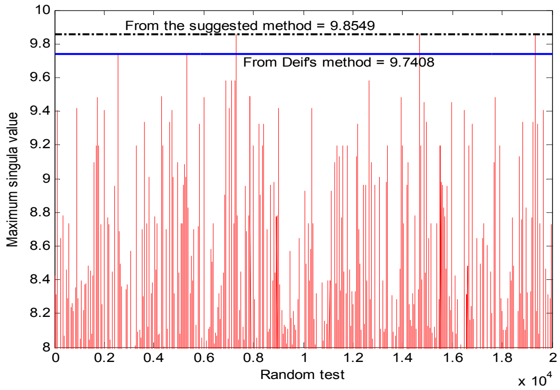

Fig. 1. Maximum singular values of randomly-selected matrices and the calculated maximum singular values from the suggested method (dashed-dot line) and from Deif's method (solid line).

Using the suggested method and using the formula of [16],[2] we find the maximum singular value of A^I as 13.937086582367124 and the minimum singular value of A^I as 0.11471992682999. Fig. 2 shows the Monte-Carlo type random test. As shown in these figures, both the calculated maximum and minimum singular values provide the upper and lower boundaries of the maximum and minimum singular values of the randomly-selected plants A, $A \in A^I$.

D. Example-4: Practical importance in a control application

In this example, we use the maximum singular value of an interval matrix for investigating the transient performance of a discrete-time uncertain system. From the following linear-time-invariant interval uncertain system

$$x_{k+1} = Ax_k, \ \ A \in A^I, \tag{10}$$

if the maximum spectral radius of A^I is less than 1, then the system is considered robust stable, which is the popular stability concept. In other words, if $\max\{\rho(A), \ \forall A \in A^I\} < 1$, where $\rho(A)$ is the spectral radius of A, then the system converges to zero. However, since only the asymptotical stability is guaranteed, there could be a big overshoot during the transient. Whereas, from the following relationship:

$$\|x_{k+1}\| = \|Ax_k\| \ \ \leq \ \ \|A\|\|x_k\|, \tag{11}$$

if $\|A\| < 1$ for all $A \in A^I$, then the state x_k will be monotonically convergent (no-overshoot) to zero (i.e., $\|x_{k+1}\| < \|x_k\|$) in a 2-norm topology, which is a more desirable convergence property than the asymptotical stability [17], [18]. Let us use the following example for a demonstration purpose:

$$x_{k+1} = Ax_k, \tag{12}$$

where

$$A \in A^I = \begin{bmatrix} [0.7650, 0.9350] & [0.0, 0.0] \\ [-6.0500, -4.9500] & [0.6700, 0.8250] \end{bmatrix}.$$

[2]Under the condition that A^I is regular, equation 1.3 and equation 1.4 of [16] can be directly used.

Since A^I is a lower triangular matrix, the maximum spectral radius is 0.9350. Hence, the system is considered robustly asymptotical stable. However, using the suggested algorithm in this paper, the maximum singular value is calculated as 6.1759. Hence, the monotonic convergent is not guaranteed. Fig. 3 shows the transients of the randomly-selected plants $A \in A^I$. As shown in these plots, it is clearly observed that there are overshoots during the transient responses although all plants converge to zero eventually as k increases.

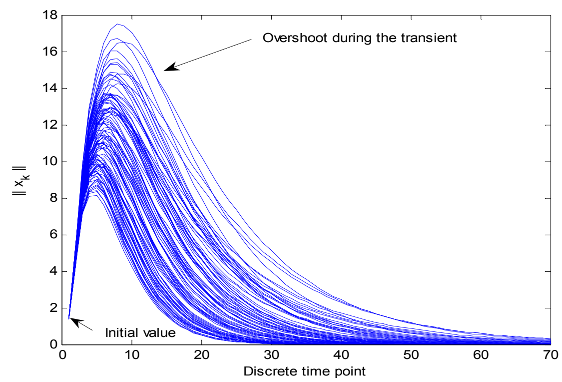

Fig. 3. Transient responses of randomly-selected plants $A \in A^I$.

Let us consider another example:

$$A^I = \begin{bmatrix} [0.8075, 0.8925] & [0.0, 0.0] \\ [-0.2625, -0.2375] & [0.7125, 0.7875] \end{bmatrix}.$$

Since A^I is a lower triangular matrix, the maximum spectral radius is found as 0.8925. Using the suggested algorithm in this paper, the maximum singular value is found as 0.9916. Thus, the system will be monotonic convergent (no-overshoot). Hence, we expect no overshoot during the transient response. Fig. 4 shows the transient responses of the randomly-selected plants. From this figure, we conclude that when the maximum singular value of an interval uncertain system is less than 1, the robustly monotonic convergence of the system is guaranteed.

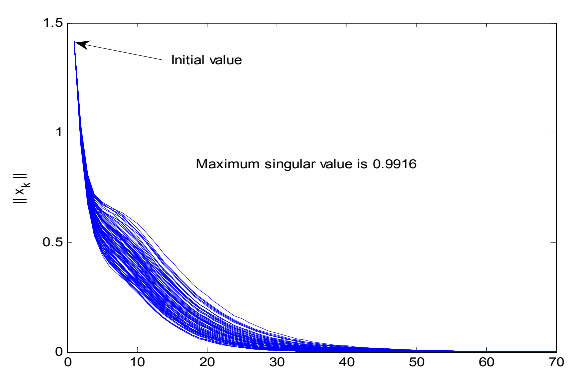

Fig. 4. Transient responses of randomly-selected plants $A \in A^I$.

Remark 5.1: In control system, a big overshoot during the transient is not acceptable because the actuator may not provide enough control force to the control system. Thus,

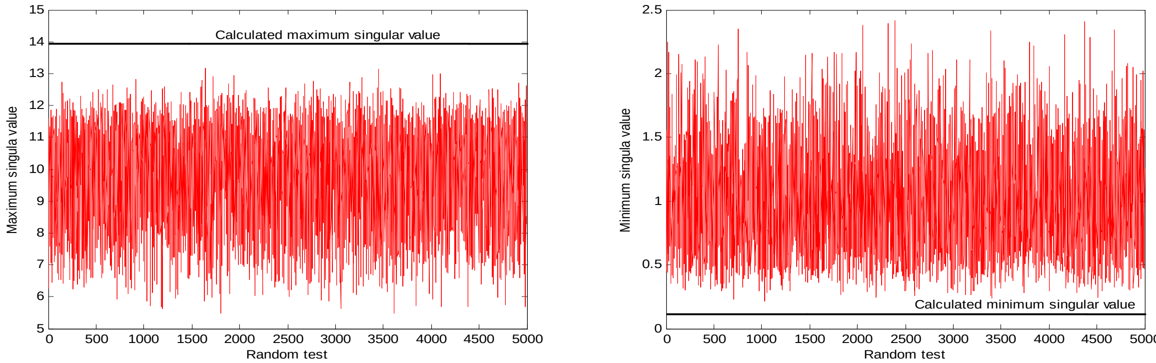

Fig. 2. Left: The maximum singular values of randomly-selected plants and the calculated maximum singular value. Right: The minimum singular values of randomly-selected plants and the calculated minimum singular value.

if possible, it is desirable to guarantee the monotonic convergence; hence finding the maximum singular value of an uncertain system matrix is practically important.

VI. Conclusions

In this paper, an algorithm for calculating the maximum singular value of a general square/non-square interval matrix was suggested. Using the existing result [13], which was developed based on perturbation under some restrictive assumptions, we verified that the proposed method can calculate the maximum singular value accurately. Furthermore, from a created example, we have shown that the existing method does not find the maximum singular value in some cases while the suggested method finds the maximum singular value without an exception. Practical importance of the maximum singular value of an uncertain system was also illustrated through an example. In authors' best knowledge, this paper presented a solution for the maximum singular value of an interval matrix, which is a fundamental question in the robust control system, for the first time.

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