ABSTRACT

This study combines fractional order control with linear quadratic regulator (LQR) for optimal robust control of a simple civil structure. As a first attempt, the purpose of this paper is to demonstrate that, when fractional damping is introduced, additional benefits can be obtained over the best traditional control method. The control problem of this paper can be used as a simple benchmark example to test new control ideas before applying to more complicated models.

Keywords: Fractional calculus, fractional order control, fractional damping, structure control, LQR.

Introduction

Today, mitigating structural responses against natural hazards like earthquakes and strong winds has become one of the most challenging topics in structural engineering. Much research has been done on both control devices to be implemented as structural elements and control algorithms to be applied to those devices to enhance the performance of the structure. Because of their simplicity and ease of use, of all the algorithms proposed for civil engineering structures, linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) algorithms have become very popular. Indeed, they are usually used as a baseline for evaluation of other control schemes.

An overview of the applications of the LQR method in seismic excited structures has been given by Soong [1]. Yang et al. [2] proposed a scheme to include the effect of acceleration response in the control gain. This is achieved by adding a weighted acceleration component to the performance index and finding new gains in terms of state space variables. The effectiveness of an Instantaneous Optimal Controller applied to a 10-story steel building frame was investigated by Chang et al. [3]. Ankireddi et al. [4] implemented LQG controllers to control wind excited tall buildings. Controller parameters were obtained through optimization of a multi objective performance criterion in which the root mean square (RMS) response of the subjected structure and the control force were constrained to be less than some prescribed values due to practical issues. Guoping et al. [5] proposed the use of an optimal control method for seismic excited linear structures considering time delays by transforming the equations into discrete time form. The optimal controller gain was obtained directly from the time delay differential equation (DDE), and can therefore be available for the case of a large time delay. The H2/LQG method was implemented in a control scheme by Ramallo et al. [6] to evaluate semi-active con-
control of a base-isolated building relative to passive isolation. To enforce the dissipation requirement for the semi-active device, a clipping secondary controller was used to filter the output of the H2/LQG controller. Simulation results for seismic-excited structures showed that smart dampers controlled by the H2/LQG algorithm can provide superior protection from a wide range of ground motions compared to the passive designs [6]. Adeli and Kim [7] presented a hybrid feedback-least mean square algorithm for control of structures through integration of the LQR or LQG algorithm and the filtered-x LMS method. Wang [8] introduced an LQG-α controller, which considers robustness and extends the LQG control design method with a relative stability and an adjustable gain parameter. The simulations of the controller on both wind and earthquake-excited buildings for some perturbations of the stiffness parameter k led to good performance.

In this paper, a fractional order controller is implemented in conjunction with the LQR algorithm on a two story shear building with actuators at each story (fully actuated case). First, a feedback controller is designed with the LQR method and the parameters in the weighting matrices, Q and R, are chosen through optimization to reach the best performance. A filtered white noise signal is used as the input excitation in the design phase. Four combinations of FOC and LQR are simulated. To compare the performance of combined LQR-FOC methods with the traditional LQR, these controllers with their optimal parameters are subjected to several existing ground motions. The results obtained demonstrate a considerable achievement in attenuating structural response.

The Simple Benchmark Civil Structure Model

The deformation response \( q \) of structural systems to ground acceleration \( \ddot{q}_g \) can be shown by the following system of equations:

\[
M\dddot{q} + C\dot{q} + Kq = Eu - ML\ddot{q}_g \tag{1}
\]

where \( M, C, \) and \( K \) are mass, damping, and stiffness matrices, respectively. \( E \) and \( I \) are influence vectors (or matrices) due to the applied control force \( u \) and the earthquake acceleration \( \ddot{q}_g \), respectively. The state-space representation of the above equation is

\[
\dot{x} = Ax + Bu + H\ddot{q}_g \tag{2}
\]

where the state vector is \( x = [q^T, \dot{q}^T]^T \), and

\[
A = \begin{bmatrix}
0_{n\times n} & I_{n\times n} \\
-M^{-1}K & -M^{-1}C
\end{bmatrix} \quad B = \begin{bmatrix}
0_{n\times m} \\
M^{-1}E
\end{bmatrix} \quad H = \begin{bmatrix}
0_{n\times 1} \\
-I
\end{bmatrix} \tag{3}
\]

In the above matrices, \( n \) denotes the system dimension (number of degrees of freedom) and \( m \) is the number of actuators. Total floor accelerations and relative story drifts are of concern for controlling the structure. Based on (1), these parameters are defined in terms of state variables and control inputs:

\[
q_d = \Delta q, \quad \Delta = \begin{bmatrix}
1 \\
-1 \\
\vdots \\
-1
\end{bmatrix}_{n\times n} \tag{4}
\]

and

\[
\ddot{q}_i = \ddot{q} + l\ddot{q}_g = -M^{-1}C\ddot{q} - M^{-1}Kq + M^{-1}Eu \tag{5}
\]

where \( q_d \) and \( \ddot{q}_i \) represent story drifts and total accelerations, respectively. Equations (4) and (5) are converted to state space representation for outputs, leading to the following simple relationship

\[
z = \begin{bmatrix}
q_d \\
\ddot{q}_i
\end{bmatrix} = C_zx + D_zu \tag{6}
\]

with \( C_z \) and \( D_z \) defined as

\[
C_z = \begin{bmatrix}
\Delta \\
-M^{-1}K & -M^{-1}C
\end{bmatrix} \quad D_z = \begin{bmatrix}
0 \\
M^{-1}E
\end{bmatrix} \tag{7}
\]

The Baseline Controller: Weight Optimized LQR

A commonly used performance index for optimal controllers has the form

\[
J = \int_0^t J_1 dt, \quad J_1 = x^TQx + u^TRu + 2x^TRu + 2x^TNu \tag{8}
\]

where \( R \) is an \( m \times m \) positive definite matrix and \( Q \) is a \( 2n \times 2n \) weighting matrix such that \( Q = NR^{-1}N^T \) is semi-positive definite. To control story drifts and accelerations defined in the output \( z \) (6) instead of state variables \( x \), a performance index aimed
at attenuating $z$ and $u$ is defined:

$$J'_f = z^T Q z + u^T R z u.$$  \hfill (9)

Using the method proposed by Yang et al. [2] leads to the following matrices for (8)

$$Q = C_z^T Q_z C_z, \quad R = D_z^T Q_z D_z + R_z, \quad N = C_z^T Q_z D_z$$  \hfill (10)

where $Q_z$ and $R_z$ are gain matrices defined for output response and control force. Using the standard linear quadratic (LQR) design with Matlab, we obtain the following full state feedback control law

$$u = -K_{LQR}x$$  \hfill (11)

where $K_{LQR}$ is the optimal feedback gain matrix obtained using $[K, S, E] = \text{LQR} \left( SYS, Q, R, N \right)$ in Matlab.

In this work, we wish to establish an optimal baseline performance for comparison to other control schemes. Therefore, an additional parameter optimization procedure is applied to search for a best set of weighting matrices $Q$ and $R$. To simplify the case, diagonal structures of $Q$ and $R$ are assumed. Henceforth, this baseline controller will be referred to as “weight-optimized LQR controller.”

### The Proposed Fractional Order Control Scheme

#### Basic Idea and Definitions

In this paper, we propose to include fractional derivative or integral of the state $x$ in the feedback control law similar to (11):

$$u = -K_{LQR}x + K_{FOC} \frac{d^\alpha x}{dr^\alpha}$$  \hfill (12)

where $K_{FOC}$ is the gain matrix to be found using optimization procedures and $\frac{d^\alpha x}{dr^\alpha}$ is defined as follows (Caputo definition, [9, 10]):

$$\frac{d^\alpha x(t)}{dr^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{x^{(n)}(\tau)}{(t - \tau)^{\alpha - 1 - n}} d\tau$$  \hfill (13)

where $n$ is an integer satisfying $n - 1 < \alpha \leq n$ and $\Gamma$ is the Euler’s Gamma function. The fractional order $\alpha$, a real number such that $\alpha \in (-1, 1)$, will be explored also. Four variants of the FOC-LQR scheme in (12) considered in this paper are as follows:

1) $K_{LQR} = 0$, $K_{FOC} = -K_{WOLQR}$

$$u = -K_{WOLQR} \frac{d^\alpha x}{dr^\alpha}$$  \hfill (14)

2) $K_{LQR} = 0$, $K_{FOC}$: from optimization

   (a) same $\alpha$ for all states

   $$u = K_{FOC} \frac{d^\alpha x_i}{dr^\alpha}$$  \hfill (15)

   (b) different $\alpha$ for each state

   $$u = K_{FOC} \frac{d^\alpha x_i}{dr^\alpha}$$  \hfill (16)

3) $K_{LQR} = K_{WOLQR}$, $K_{FOC}$: from optimization, (different $\alpha$ for each state)

$$u = -K_{WOLQR}x + K_{FOC} \frac{d^\alpha x_i}{dr^\alpha}$$  \hfill (17)

where $K_{WOLQR}$ denotes the gain matrix of the weight optimized LQR. Further explanation of the above cases is given in the following sections.

After Newton and Leibniz discovered calculus in the 17th century, fractional-order calculus has been studied as an alternative in mathematics [11, 12]. As claimed in [13], fractional order calculus will play an important role in smart mechatronic and biological systems. Recently, in control society, fractional order dynamic systems and controls have received increasing attention [14–18]. Pioneering works in applying fractional calculus in dynamic systems and controls and the recent developments can be found in [19–25]. For more detailed explanation about the fractional dynamics and control, refer to [26, 27].

Clearly, for closed-loop control systems, there are four situations: 1) IO (integer order) plant with IO controller; 2) IO plant with FO (fractional order) controller; 3) FO plant with IO controller and 4) FO plant with FO controller. In control practice, the fractional-order controller is more common, because the plant model may have already been obtained as an integer order model in the classical sense. From an engineering point of view, improving or optimizing performance is the major concern [28]. Hence, our objective is to apply the fractional-order control (FOC) to enhance the (integer order) dynamic system control performance [23, 28].
Oustaloup’s Approximation Algorithm

The approximation algorithm presented by Oustaloup [29] is widely used. In this case, a frequency band of interest is considered, within which the frequency domain responses are fit by a bank of integer order filters to the fractional order derivative. Suppose that the frequency range to be fit is given by \([\omega_a, \omega_b]\), and the term \(s/\omega_u\) can be substituted with

\[
C_0 \frac{1 + s/\omega_b}{1 + s/\omega_h}
\]  

(18)

where

\[
\sqrt{\omega_b \omega_h} = \omega_u
\]  

(19)

and

\[
C_0 = \frac{\omega_h}{\omega_u} = \frac{\omega_h}{\omega_h}
\]  

(20)

Using the zigzag piecewise approximation in the Bode plot, Oustaloup’s approximation model to a fractional order differentiator \(s^\alpha\) can be written as

\[
\hat{H}(s) = \left(\frac{\omega_u}{\omega_h}\right)^\alpha \prod_{k=-N}^{N} \frac{1 + s/\omega'_k}{s/\omega_k}
\]  

(21)

where

\[
\omega'_k = \omega_b \left(\frac{\omega_h}{\omega_h}\right)^{\frac{kN + 1/2}{2N + 1}}
\]  

(22)

and

\[
\omega_k = \omega_b \left(\frac{\omega_h}{\omega_h}\right)^{\frac{kN + 1/2}{2N + 1}}
\]  

(23)

are respectively the zeros and poles of rank \(k\). Here, \(2N + 1\) is the total number of zeros or poles.

The above continuous-time approximation has been enhanced and modified in [30] and a Simulink block is provided and illustrated in [31]. As a side remark, other finite integer order approximation schemes in discrete-time form are available [32]. In this paper, we use the Simulink block for \(s^\alpha\) based on the modified Oustaloup’s approximation [30] from [31].

### Table 1. Structural parameters

<table>
<thead>
<tr>
<th>Floor Masses (kg)</th>
<th>Stiffness (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1 = 6000)</td>
<td>(k_1 = 6141)</td>
</tr>
<tr>
<td>(m_2 = 4000)</td>
<td>(k_2 = 3509)</td>
</tr>
</tbody>
</table>

### Numerical Example

The structure considered is a two story shear building excited by ground motions at the base level. A schematic view of the structure together with its degrees of freedom is shown in Fig. 1.

Figure 1. Schematic view of a simple 2 story structure

![Schematic view of a simple 2 story structure](image)

Structural mass and stiffness are given in Table 1. Natural periods of the building are 0.3 and 0.14 seconds for the first and second modes, respectively.

Rayleigh damping is applied based on 2% damping in each mode. The magnitude of control force applied to the structure is bounded to \(\pm 20\) kN.

### LQR Weight Optimization Process

One of the biggest issues in implementing optimal controllers is selecting the best gain parameters. The control gain obtained from the LQR algorithm is completely dependent on the objective function defined in (8). Through this index, designers can emphasize attenuating the structural responses that are of greatest concern. While this index provides intuition to select the
pattern for gain matrices, it definitely will not result in an optimal design. Furthermore, the force capacities of both actuators (to apply force) and connections (to which force is exerted) are limited, and as a consequence, the calculated input force should be bounded. This issue also increases the complexity of choosing gain matrices. To solve this problem, a performance criterion be bounded. This issue also increases the complexity of choosing gain matrices. To solve this problem, a performance criterion different from the one introduced in (8) is proposed:

\[ PI = \beta_1 \sum_i \frac{\text{RMS}(z_c)}{\text{RMS}(z_0)} + \beta_2 \sum_i \frac{\max|z_c|}{\max|z_0|} \]  

where \( z_c \) and \( z_0 \) are the output of the controlled and uncontrolled cases, respectively. The first component emphasizes the mitigation of the root mean square response and the second component the peak response. The parameters \( \beta_1 \) and \( \beta_2 \) in the function give designers the ability to specialize the performance index for specific purposes. For instance, if the aim is to resist against extreme events like earthquakes, peak response rather than RMS response should be reduced or minimized to prevent collapse. However, in windy zones where the occupants comfort level is of greater concern, RMS response would govern design requirements and emphasis can be placed on the first component of the performance index. The objective function defined above (24) is used in an optimization process to find the most appropriate weighting parameters in (8).

The nature of earthquakes is stochastic and a controller designed for only one earthquake record may not give good performance during other earthquakes. To account for this property of the excitation, 64 artificially generated earthquake records are used in the optimization procedure. To produce these records, white noise signals were passed through a Kanai-Tajimi filter [33]. The MATLAB SIMULINK package and Optimization Toolbox were used to simulate the building, controller, and earthquake records. Figure 2 shows the optimization model in SIMULINK.

In this paper \( \beta_1 \) and \( \beta_2 \) are assumed to be 1 and 2 respectively. The optimization process led to \( Q_c = \text{diag}([20.919, 60.993, 8.216e-7, 48.427]) \) and \( R_c = \text{diag}([6.088e-7, 9.9783e-7]) \). For Case (2a), \( K_{\text{FOC}} \) is defined from optimization rather than assumed as \( K_{\text{WOLQR}} \) (15). Since \( K_{\text{FOC}} \) is \( 2 \times 4 \), 9 parameters must be found in an optimization process. Simulation results show large improvements in responses where \( J \) is reduced by 36% with respect to the optimized LQR method. In particular, story drifts are reduced significantly. Story drifts in the first and second stories are reduced by 77% and 91% for RMS drift and 43% and 76% for peak drift, respectively (Table 2). Although mitigation of acceleration response is not as significant as for drift, a considerable reduction is still seen. Accelerations in the first and second floor are reduced by 13% and 10% for RMS acceleration, and 2% and 10% for peak acceleration, respectively (Table 2).

In Case (1) of combined FOC and LQR, the controller is assumed to have only the fractional part, i.e. \( K_{\text{LQR}} = 0 \), and the input is derived through (14). The gain matrix for \( \alpha \)-order state variables, \( K_{\text{FOC}} \), is the optimized weight LQR gain matrix \( K_{\text{WOLQR}} \) and the only parameter to be identified is the fractional order, \( \alpha \). Peak and RMS responses of the structure for different values of \( \alpha \) normalized with respect to the comparable optimized LQR responses are shown in Fig. 3. The value of \( \alpha \) that minimizes the relative response varies for different types of response measures and outputs but is generally somewhere between 0 and 0.2 (Fig. 3). Using the objective index in (24), \( \alpha_{\text{opt}} \) is found to be 0.05. As can be seen, this controller does not result in significant reduction in response compared to the optimized LQR method. This result could have been predicted beforehand, because the only parameter optimized is the order, \( \alpha \), and the weight optimized LQR gain matrix definitely is not the best choice for this case. Thus, reduction in the \( J \) factor with respect to the optimized LQR is only 2% (Table 2).

![Figure 3. Relative response of the case 1 controller](image-url)
for Case (2b), different values of $\alpha$ are assigned for each state variable; hence three extra variables are added to the system. Case (2b) results in relatively similar performance to the previous Case (2a) where the same order was used for all state variables. The objective index $J$ is reduced by 37% relative to the optimized LQR method, compared to 36% for the controller of Case (2a) (Table 2). Therefore, the increased computational complexity to identify different orders for the state variables brings only marginal benefit.

In case (3), a fractional order controller is added to the weight optimized LQR, i.e. $K_{LQR} = K_{WOLQR}$, and the gain matrix for the fractional part and the orders of state variables are found through optimization. In this case, $J$ is reduced by 30% with respect to the weight optimized LQR method, showing less improvement compared to case (2a) and case (2b).

Simulation Results for Real Ground Motions

To observe the performance of different control types introduced here in real world situations, the building structure is subjected to three previously recorded ground motions: 1940 El Centro at Imperial Valley (PGA 0.3129 g), 1995 Kobe at Japanese Meteorological Agency (PGA 0.8213 g), and 1994 Northridge at Sylmar (PGA 0.8433 g). In this part, gain matrices and state orders, found from optimization processes in the previous section, are applied to the controllers. Simulation results are presented in Table 3.

The response pattern for the realistic ground motions is similar to that of the optimization part. This generally verifies the procedure conducted to obtain the best parameters for the gain matrix and/or the order of state variables. According to the results presented in Table 2 based on the 64 artificially generated ground motions, the objective index and almost all response measures for case (3) are worse than the corresponding results in case (2a) and case (2b). This trend is also reflected in structural responses for El centro earthquake in Table 3 where the performance index is by 36% larger than the one obtained in case (2b), the controller which gives the best response in El centro earthquake. However, response reductions in Kobe and Northridge motions are better than the related ones achieved in case (2a), the most appropriate controller for the mentioned ground excitations.

Results presented in Table 3 also indicate the performance achieved during El centro earthquake for all types of controllers is much better than the performance obtained in Kobe and Northridge earthquakes. One of the factors having influence on the performance of control systems is the acceleration records used in the optimization process. Although artificial records generated by Kanai-Tajimi filter have relatively close power spectral density curves to real ground motions, their pattern in time domain could be largely different. In this sense, El centro earthquake record is seen to be more close to the generated records than the other two ones. The other factor that can affect the performance is the PGA of excitations. The larger the peak ground acceleration, the higher amplitude of input force is required. As mentioned in previous sections, an upper and lower bound is placed on the control force to account for actuators and joints capacities. These bounds prevent controllers to apply desired forces leading to substantial degradation of the efficiency of control systems in cases where the difference between the desired and applied forces are considerable. The effect of this factor on the structural response for Kobe and Northridge earthquakes with PGAs roughly 2.5 times the PGA of El centro ground motion is more considerable.
Table 2. Root mean squares (RMS) and peak structural responses with their standard deviations for the optimization part.

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>Response</th>
<th>Drift (cm)</th>
<th>Acceleration (g)</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measure</td>
<td>1st floor (SD)</td>
<td>2nd floor (SD)</td>
<td>1st floor (SD)</td>
</tr>
<tr>
<td>Uncontrolled</td>
<td>RMS</td>
<td>0.985 (0.148)</td>
<td>0.966 (0.149)</td>
<td>0.464 (0.064)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>2.976 (0.412)</td>
<td>2.908 (0.425)</td>
<td>1.444 (0.188)</td>
</tr>
<tr>
<td>LQR Controlled</td>
<td>RMS</td>
<td>0.318 (0.019)</td>
<td>0.259 (0.017)</td>
<td>0.179 (0.010)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>1.086 (0.160)</td>
<td>0.896 (0.148)</td>
<td>0.622 (0.080)</td>
</tr>
<tr>
<td>Case (1)</td>
<td>RMS</td>
<td>0.304 (0.019)</td>
<td>0.250 (0.019)</td>
<td>0.175 (0.010)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>1.061 (0.174)</td>
<td>0.914 (0.171)</td>
<td>0.626 (0.083)</td>
</tr>
<tr>
<td>Case (2a)</td>
<td>RMS</td>
<td>0.074 (0.014)</td>
<td>0.024 (0.006)</td>
<td>0.155 (0.010)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>0.615 (0.143)</td>
<td>0.236 (0.105)</td>
<td>0.608 (0.105)</td>
</tr>
<tr>
<td>Case (2b)</td>
<td>RMS</td>
<td>0.075 (0.015)</td>
<td>0.022 (0.006)</td>
<td>0.155 (0.010)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>0.616 (0.144)</td>
<td>0.213 (0.113)</td>
<td>0.605 (0.104)</td>
</tr>
<tr>
<td>Case (3)†</td>
<td>RMS</td>
<td>0.108 (0.008)</td>
<td>0.059 (0.006)</td>
<td>0.175 (0.009)</td>
</tr>
<tr>
<td></td>
<td>Peak</td>
<td>0.552 (0.092)</td>
<td>0.353 (0.092)</td>
<td>0.658 (0.067)</td>
</tr>
</tbody>
</table>

*SD stands for standard deviation.
†Results are due to early termination of running.

Conclusion and Future Research Efforts

The application of fractional order controllers in conjunction with the LQR method has been introduced in this paper. Several combinations of FOC and LQR were considered and subjected to optimization processes to find the most appropriate parameters. 64 artificially generated earthquakes were used to optimize the controller gains. Simulation results demonstrated that introducing FOC into the LQR method led to a great advance in attenuating the response over optimized LQR alone. The best performance was produced when a single fractional order was assigned to all state variables and the gain matrix was found from optimization. Considering distinct fractional orders for each state variable did not appreciably improve the performance, and in some cases the structural response even became worse. Simulating the system with actual recorded ground motions led to the same trends for response attenuation, implying that the optimization process works well.

To develop a simple model with which to apply proposed controller, the structure has been assumed to be fully observable. However, this assumption is far from realistic and considering noise effects will degrade the efficiency of controllers. Next, the performance of the proposed controllers should be assessed in a more realistic setting, where observer-based controllers are designed based on a filtering the noise measurements.

REFERENCES


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Table 3. Root mean squares (RMS) and peak structural responses.

| Controller Type | Earthquake | Drift (cm) | | | | | | Acceleration (g) | | | | | |
|-----------------|------------|------------|---|---|---|---|---|---|---|---|---|
|                 |            | 1st floor  | 2nd floor | 1st floor | 2nd floor | J |
|                 |            | RMS | Peak | RMS | Peak | RMS | Peak | RMS | Peak | RMS | Peak |
| Uncontrolled    | Elcentro   | 0.287 | 1.206 | 0.287 | 1.317 | 0.143 | 0.771 | 0.256 | 1.182 | 12 |
|                 | Kobe       | 0.533 | 2.820 | 0.493 | 2.469 | 0.269 | 1.472 | 0.441 | 2.207 | 12 |
|                 | Northridge | 0.829 | 4.695 | 0.810 | 3.873 | 0.391 | 2.598 | 0.725 | 3.462 | 12 |
| LQR controlled  | Elcentro   | 0.108 | 0.676 | 0.084 | 0.494 | 0.062 | 0.366 | 0.064 | 0.367 | 4.795 |
|                 | Kobe       | 0.347 | 2.223 | 0.301 | 1.843 | 0.183 | 1.191 | 0.227 | 1.536 | 8.534 |
|                 | Northridge | 0.314 | 3.264 | 0.288 | 2.832 | 0.161 | 1.755 | 0.225 | 2.363 | 7.026 |
| Case (1)        | Elcentro   | 0.107 | 0.695 | 0.084 | 0.523 | 0.061 | 0.440 | 0.060 | 0.367 | 5.035 |
|                 | Kobe       | 0.349 | 2.287 | 0.313 | 1.985 | 0.182 | 1.274 | 0.226 | 1.547 | 8.841 |
|                 | Northridge | 0.324 | 3.368 | 0.306 | 3.003 | 0.164 | 1.849 | 0.233 | 2.404 | 7.306 |
| Case (2a)       | Elcentro   | 0.011 | 0.228 | 0.004 | 0.088 | 0.055 | 0.360 | 0.054 | 0.351 | 2.690 |
|                 | Kobe       | 0.172 | 1.148 | 0.093 | 0.981 | 0.170 | 1.221 | 0.185 | 1.381 | 6.084 |
|                 | Northridge | 0.164 | 2.196 | 0.115 | 1.637 | 0.142 | 1.497 | 0.166 | 1.972 | 5.004 |
| Case (2b)       | Elcentro   | 0.011 | 0.234 | 0.004 | 0.071 | 0.055 | 0.360 | 0.055 | 0.336 | 2.645 |
|                 | Kobe       | 0.180 | 1.246 | 0.099 | 1.053 | 0.175 | 1.407 | 0.194 | 1.450 | 6.594 |
|                 | Northridge | 0.168 | 2.185 | 0.117 | 1.723 | 0.144 | 1.749 | 0.169 | 2.054 | 5.303 |
| Case (3)+       | Elcentro   | 0.037 | 0.278 | 0.020 | 0.256 | 0.069 | 0.492 | 0.058 | 0.331 | 3.594 |
|                 | Kobe       | 0.168 | 1.194 | 0.109 | 0.977 | 0.171 | 1.050 | 0.174 | 1.381 | 5.881 |
|                 | Northridge | 0.166 | 2.267 | 0.124 | 1.531 | 0.147 | 1.545 | 0.160 | 1.868 | 4.977 |

*SD stands for standard deviation.
†Results are due to early termination of running.