Stability Analysis and Control of Repetitive Trajectory Systems in the State-Domain: Roller Coaster Application

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Abstract—This paper discusses the concepts of stability and asymptotic stability for state-dependent repetitive trajectory systems. An adaptive compensation method is proposed to compensate for disturbances that are a function of the system state. It will be shown that Lyapunov stability analysis can be performed in the state-domain if certain conditions are satisfied. The main theoretical contribution of the paper is to show that stability analysis can be carried out not only along the time axis but also in the state domain and that state-dependent disturbances or uncertainty can be successfully rejected. The motion dynamics of a roller coaster system is simulated to illustrate the practical importance of the proposed compensation method.

Index Terms—State-domain; Lyapunov stability; Stability on a trajectory; Roller coaster

I. INTRODUCTION

Disturbance compensation is important in many engineering problems, including servo-motor control, speed control of rotary systems, mobile motion control, vehicle trajectory control, etc. Though in general disturbances acting on systems can be a function of states or a time-dependent periodic external force or torque, most previous efforts have been devoted to the time-dependent problems, such as the adaptive disturbance compensation schemes in [2], [3], the hardware configuration-based disturbance compensation strategy in [4], or the time-periodic adaptive controller introduced in [5] and applied to the adaptive friction compensation problem in [6]. However, in practice, state-dependent external disturbances exist in a variety of engineering problems. For example, in [7], [8], the engine crankshaft speed pulsation was expressed as a Fourier series expansion, which is a function of position. In [9] it was shown that the network congestion intensity can be a function of the transmission rate, which is a state of the system. In [10], the external disturbance of the satellite was modelled as a function of the position. In [4], the tire/road contact friction was represented as a function of the system state variable. Yet, relatively very few research efforts have been devoted to the state-dependent adaptive disturbance compensation problem. From a literature search, two different approaches are noticeable for nonlinear state-dependent adaptive disturbance compensation. The first approach is related with mechanical nonlinearities such as friction. The friction force depends on the velocity or position. With regard to friction force compensation, after Friedland, et al.’s works in [11], [12], [13], several adaptive friction compensation controllers have been designed in [14], [15], [16]. However, these efforts were restricted to Coulomb friction, which is a constant disturbance with a discontinuity at the zero velocity. The second approach is related to state-dependent eccentricity [17], [18], [19] and servo-motor speed control [20]. In [18], an adaptive eccentricity compensation problem, which is a position-dependent oscillatory disturbance, was solved. However, in [18], the disturbance was specified as the following form:

\[ d(x) = \Lambda \cos(\omega x + \Phi), \]

where \( \Lambda, \omega, \) and \( \Phi \) are unknown. In [20], the speed of the servo-motor was controlled with an unknown disturbance (just position-dependent) using iterative learning control. For visual examples and more detailed explanation about the position-dependent disturbance, refer to [18].

Recently, it is shown in [1] that a state-dependent periodic disturbance in a general form can be compensated for by making use of the state-dependent periodicity. This paper provides a more generalized state-dependent stability concept based on [1]. In particularly, we discuss concepts of stability and asymptotic stability on a trajectory to compensate for disturbances that are a function of the system state. It will be shown that Lyapunov stability analysis can be performed in the state-domain if some conditions are satisfied. Thus, the contribution of this paper over [1] is to provide a generalized stability analysis in the state-domain. Also, as an application of [1], a simplified motion model of a roller coaster system is used to show that a state-dependent disturbance in the roller coaster can be successfully compensated for based on our ideas of stability analysis and control of repetitive trajectory systems.

The paper is organized as follows. In Section II, a generalized stability concept for state-dependent trajectory systems is provided. In Section III, some preliminary required assumptions are provided to make the problem clear. In Section IV, stability analysis in the state-domain...
is performed. In Section V, a motion model of a roller coaster system is used to illustrate the validity and practical importance of the proposed method. Conclusions are given in Section VI.

II. PROBLEM DEFINITIONS

In this paper, we will consider the following simple servo-control system:

\[
\begin{align*}
\dot{x}(t) &= v(t) \\
\dot{v}(t) &= a(x(t)) + u(t),
\end{align*}
\]

where \( x \) is the position, such as angular displacement, \( a(x(t)) \) is an unknown state (i.e., \( x \))-dependent disturbance, \( v \) is the speed, and \( u \) is the control input such as torque. The state-dependent disturbance \( a(x(t)) \) can be any function of state \( x \), which is dependent on the time \( t \). Thus, eventually the disturbance \( a \) is also a function of time \( t \). However, in our problem definition, the disturbance always has the same value whenever the state has a given value, no matter what the value of time is. That is, suppose \( x(t_a) = x(t_b) \), with \( t_a \neq t_b \). Then \( a(x(t_a)) = a(x(t_b)) \). For illustration, consider a roller coaster as shown in Fig. 1. The roller coaster has several different state-dependent external disturbances. The roller coaster experiences a gravity force depending on the position on a trajectory and it has a friction force which is also dependent on the position and/or speed.\(^1\) The roller coaster is controlled to move forward, but it also rises and falls on a fixed track, which makes the control task a state-dependent disturbance compensation problem. Actually, we can see many engineering problems that have similar motion characteristics and state-dependent disturbances. For instance, rolling mills [2], satellites [10], the shear force between two contacting, rotating bodies, orbit/trail systems, and engine crankshafts [7] all have position- or speed-dependent external disturbances along fixed trajectories. In what follows, for the state-dependent stability analysis, we repeat some preliminary required definitions and assumptions from [1].

Definition 2.1: The total passed trajectory is given as:

\[
s(t) = \int_0^t |\frac{dx}{d\tau}| d\tau = \int_0^t |v(\tau)| d\tau,
\]

where \( x \) is the position, and \( v \) is the speed. In [17], \( s(x) \) was defined as the curvilinear abscissa associated with the trajectory of the relative motion. In our definition, since \( s \) is the summation of absolute position increasing along the time axis, \( s \) is a monotonically-growing signal.

Physically, the total passed trajectory has the following property:

\[
s(t_1) \geq s(t_2), \text{ if } t_1 \geq t_2.
\]

However, as shown in Fig. 2, we can consider three different trajectories. In Fig. 2, case-(a) shows that the trajectory is repetitive on a fixed track. The state is monotonically increasing and monotonically decreasing in the fixed track. Case-(b) shows a situation where the state is not monotonically decreasing (in the circled-area). Although in case-(b), the total passed trajectory is monotonically changing, the trajectory is not repetitive. Case-(c) shows a situation where the state is stationary (again in the circled-area). So, in this case, velocity \( v(t) \) could be zero and the total passed trajectory is not increasing in this stationary point. Thus, the total passed trajectory is semi-monotonically increasing in case-(c). In this paper, we restrict our attention to repetitive trajectory systems such as those depicted in case-(a) of Fig. 2.

Next, we provide some formal stability definitions for the repetitive trajectory systems.

\(^1\)http://www.myphysicslab.com/RollerSimple.html
Definition 2.2: Concepts of equilibrium points, stability, and asymptotical stability of a state-dependent nonlinear repetitive trajectory systems are defined based on Definition 4.4 of [21]. Given a state-dependent nonlinear repetitive trajectory system
\[ \dot{x} = f(s, x), \]
- origin \( x = 0 \) is an equilibrium point for (3) if \( f(s, 0) = 0, \forall s \geq 0. \)
- stable if, for each \( \epsilon > 0 \), there is a \( \delta = \delta(s_0, \epsilon) > 0 \) such that
  \[ \|x(s_0)\| < \delta \Rightarrow \|x(s)\| < \epsilon, \forall s > s_0 \geq 0. \]
- asymptotically stable if it is stable and there is a positive constant \( c = c(s_0) > 0 \) such that \( \|x(s)\| \to 0 \) as \( s \to \infty \), for all \( \|x(s_0)\| < c. \)

In speed tracking problems, we need to define an error such as \( e_v = v - v_d \) where \( v_d \) is a desired speed trajectory, for an equilibrium point. The equilibrium point for \( e_v \) is \( v_d \).

Definition 2.3: For non-autonomous repetitive trajectory systems (3), if \( f(s, x) \) satisfies the inequality, with \( L > 0, \)
\[ \|f(s, x) - f(s, y)\| \leq L\|x - y\| \]
for all \((s, x)\) and \((s, y)\) in some neighborhood of \((s_0, x_0)\), then the system is Lipschitz continuous.

In this paper, Lipschitz continuity can be simply written as \(|f(y) - f(x)| \leq L|y - x|\), because we consider continuity on a single state-domain \( x \).

Using the definitions given above, we can perform Lyapunov stability analysis on the \( s \)-domain (refer to [22], [23], [21] for Lyapunov functions and stability analysis on the time domain) as follows:

Definition 2.4: Let \( G \) be a set in \( \mathbb{R}^n \) and let \( G^* \) be an open set of \( \mathbb{R}^n \) containing \( \overline{G} \), the closure of \( G \). For the nonautonomous nonlinear system (3), \( V \) is a Lyapunov function on \( G \) if it is continuous and locally Lipschitzian on \( s \in [0, \infty) \times G^* \) and if

- given \( x \) in \( G \), there is a neighborhood \( N \) of \( x \) such that \( V(s, x) \) is lower bounded for all \( s \geq 0 \) and all \( x \) in \( N \cap G. \)
- \( \dot{V}(s, x) \leq -W(x) \leq 0 \) for all \( s \geq 0 \) and all \( x \) in \( G \), where \( W \) is continuous on \( G \).

Using the Lyapunov function defined in Definition 2.4, the following stability condition is immediate:

Definition 2.5: (Theorem 4.1 of [23]) If, in a ball around the equilibrium point \( x = 0 \) of (3), there exists Lyapunov function \( V(s, x) > 0 \) and \( \dot{V}(s, x) \leq 0 \), the equilibrium point is then stable.

III. Assumptions and Properties

For stability analysis on the state-domain of a repetitive trajectory system as depicted in Fig. 2-(a), we need the following assumptions (actually, it can be shown that the following assumptions are properties of a trajectory repetitive system).

Assumption 3.1: The external disturbance, \( a(s) \), is smooth and Lipschitz continuous along the trajectory \( (s) \), i.e., there exists the following smooth derivative:
\[ \nabla a = \frac{da(s)}{ds}, \forall s \in \mathbb{R}^+ \]

Assumption 3.2: The dynamic system is evolving in one direction, and \( v(t) \neq 0. \)

Note that the idea of "progress along the path" was firstly provided in [24].

Remark 3.1: With Assumption 3.2, a derivative operator can be defined as, which was also appeared in [25]:
\[ \nabla \equiv \frac{d}{ds} = \frac{1}{v(t)} \frac{d}{dt}. \]

Furthermore, since \( v(t) > 0 \), we have \( s(t_1) = s(t_2) \) iff \( t_1 = t_2. \)

In the rotating system, we have the property \( 0 \leq x < 2\pi \), while \( 0 \leq s(x) < \infty \) where \( s(x) \) was defined in Definition 2.1. Without notational confusion, we simply write \( s(x) \) by \( s \), and write \( a(s(x)) \) by \( a(s) \) on the \( s \)-domain.

It is then assumed that \( a(s) \) is Lipschitz continuous along the trajectory \( s \) and has the following periodicity:
\[ a(s) = a(s - s_p), \]
where \( s_p \) is the state-dependent period. The following relationship is given to avoid conceptual confusion:
\[ x = s - ms_p, \]
where \( m \) is the integer part of the quotient of \( s/s_p \). In a rotary system, it is reasonable to assume that \( s_p \) is fixed as \( 2\pi \) radians. Before proceeding, let us examine the relationship between \( t, s, \) and \( x. \) Given a particular time \( t \), there always exists the corresponding \( s \), because there is a bijective relationship between \( t \) and \( s \). Furthermore, given a particular \( s \), there also exists the corresponding
x such that x = s − ms_p holds. Thus, the relationships
a(t) = a(x) = a(s) and v(t) = v(x) = v(s) are true.
Similarly, when the desired velocity is given in terms of x
like v_d(x), we can always find v_d(s) and \frac{dv_d(s)}{ds} using
\begin{align}
v_d(x) = v_d(x + ms_p) = v_d(s) \quad \text{and} \quad \frac{dv_d(x)}{dx} = \frac{dv_d(s)}{ds}
\end{align}
(6)
where x and s are the particular position and total passed
trajectory corresponding to a particular time t, respectively.
When v_d(x) is given, it is supposed that v_d(s) and \frac{dv_d(s)}{ds}
are found. In this case, based on (6), we can design a speed
tracking controller for the repetitive trajectory systems using
v_d(s) and \frac{dv_d(s)}{ds} on the s domain.

IV. STABILITY ANALYSIS IN THE STATE-DOMAIN

The control objective is to track or servo the corresponding
desired velocity v_d(s) with a tracking error as small as
possible. In practice, it is also reasonable to assume that
a(s) and \dot{v}_d(s) are all bounded. The compensation
approach is summarized as follows:

- When s < s_p, we control the system to be bounded
input bounded output (in L_2-norm).
- When s ≥ s_p, we stabilize the system to track the
desired speed. By state-dependent periodic adaptation,
we can also estimate the unknown a(s).

The following notations are used:
\[ e_a = a(s) - \dot{a}(s); \quad e_v = v(s) - v_d(s) \]
First, we consider the case when s ≥ s_p. The feedback
control law is designed as:
\[ u = \dot{a}(s) + v \frac{dv_d(s)}{ds}, \quad (7) \]
where \dot{a}(s) is the adapted state-dependent disturbance. The
adaptation law is designed as
\[ \dot{a}(s) = \dot{a}(s - s_p) - K_1 e_v \frac{v(s)}{v(s)}, \quad (8) \]
where K_1 is a positive design parameter. Now, we are able
to develop the following theorems.

Theorem 4.1: If s and v(s) are measured and e_a and e_v
have been L_2-bounded when s < s_p, then as s → ∞, the
equilibrium points, e_a and e_v, are asymptotically stabilized
by the control law (7) and the adaptation law (8).

Proof: The proof can be carried out using the operator
defined in Remark 3.1. The key idea is to use parameter
variation. For a detailed explanation, see [1].

Now, we consider the case when s < s_p. In this case,
we only have to guarantee the bounded output as required
in Theorem 4.1. For this case, we use the following control
law and adaptation law:
\[ u = \dot{a}(s) - K_1 e_v \quad (9) \]
\[ \dot{a}(s) = z(s) - g(v), \quad (10) \]
where g(v) = \frac{1}{2} Kv^2 and z(s) = -K_a K e_v s, with K > 0
and K_a > 0.

Theorem 4.2: When 0 ≤ s < s_p, if a(s) is continuously
differentiable along s including at s_p, then e_a and e_v are
all L_2-bounded by the control law (9) and the adaptation
law (10).

Proof: See [1].

Remark 4.1: In Theorem 4.2, the assumption that “a(s)
is continuously differentiable along s including at s_p”
could make the proof trivial. However, this assumption is
physically valid in many engineering problems, such as
sinusoidal disturbances [18], cogging forces [26], [27],
and servo-motor control [20]. However, we also note that
this “continuously differentiable” assumption might not be
true in friction force compensation problems [6], [11], [12],
[13].

Recall that we have developed the control law and
adaptation law when s < s_p to bound e_a and e_v; then we
used the information of one past period (i.e., \dot{a}(s - s_p)) to
stabilize e_a and e_v as s → ∞ when s ≥ s_p. It is important
to emphasize that the Lyapunov analysis was performed
along the state-trajectory axis (s). Finally, the following
remark is provided for the stability on the time domain.

Remark 4.2: From Definition 2.1 and Assumption 3.2, it
is easy to show that s → ∞ if and only if t → ∞. Hence,
we conclude that e_v(t) → 0 as t → ∞ if e_v(s) → 0 as
s → ∞.

V. ROLLER COASTER SPEED CONTROL

In this section, we provide an illustrative example using a roller coaster motion model. A simple roller coaster motion equation is given in
“http://www.myphysicslab.com/RollerSimple.html” as
\[ \dot{x} = v \quad (11) \]
\[ \dot{v} = -\frac{gk(x)}{\sqrt{1 + (k(x))^2}} - b/m \cdot v, \quad (12) \]
where g is the gravity constant, k(x) is a position-dependent
disturbance, b is a constant determining the amount of damping, and m is a mass. From a control engineering
perspective, in this simulation, it is assumed that a control
force u can be applied to the roller coaster system as follows
\[ \dot{x} = v \quad (13) \]
\[ \dot{v} = -\frac{gk(x)}{\sqrt{1 + (k(x))^2}} - b/m \cdot v + u. \quad (14) \]

For simulation, the following k(x) is used
\[ k(x) = 0.1\sin(x) - 0.5\sin(x/2)^2 - 0.5\cos(2x)^2 + 0.5\cos(x/2)^2, \]
which is smooth and Lipschitz continuous as shown in Fig. 3. The desired speed and the desired acceleration are given as functions of position as:

\[ v_d(x) = 2\sin^2(x/2) + 0.5 \]  \hspace{1cm} (15)

\[ \frac{dv_d}{dx} = 2\sin(x/2)\cos(x/2), \]  \hspace{1cm} (16)

where \( x \ (0 \leq x < s_p) \) is the repetitive trajectory and \( v_d(x) \) is the desired speed trajectory that is Lipschitz continuous. The control gains are selected as \( K = 3.0, \ K_2 = 3.0 \) and \( K_1 = 100 \), respectively. It is assumed that the damping parameter is known previously. Hence, the control input \( u \) includes \( b/m \cdot \ddot{v} \), where \( \ddot{v} \) is a measured speed, to compensate for the damping term \( b/m \cdot v \), which is dependent on the speed. The gravitational constant is 9.8. Clearly, \( a(x) = \frac{gk(x)}{\sqrt{1+(k(x))^2}} = a(x + 2\pi) = \frac{gk(x+2\pi)}{\sqrt{1+(k(x+2\pi))^2}} \). In the simulation, the initial position, initial velocity and initial disturbance are assumed to be zero. Fig. 3 shows the actual state-dependent disturbance and achieved (estimated) disturbance. As the length of total passed trajectory increases, the disturbance is estimated to be the true value. Fig. 4 shows the speed tracking results where in the top subplot the dashed-line is the desired speed trajectory and the solid-line is the achieved speed trajectory, all with respect to the position and the bottom subplot shows the corresponding speed tracking error. In the top subplot of Fig. 5, the dashed-line is the desired speed trajectory and the solid-line is the achieved speed trajectory, all with respect to time. The bottom subplot of Fig. 5 shows the speed tracking error versus time. In this simulation, \( s_p \) is \( 2\pi \) radians. As shown in the figures, until \( 4\pi \) radians, there exists a relatively big speed tracking error \( (0.3 \text{ radians/second}) \). After \( 4\pi \) radians, the desired speed is tracked well. From the comparison between Fig. 4 and Fig. 5, we observe that the position-dependent speed trajectory is different from the time-dependent speed trajectory. The top of Fig. 6 shows the adaptive compensation input signal versus the position travelled in radians while the bottom subplot of Fig. 6 shows the same input signal versus time. From Fig. 4 and Fig. 5, we observe that the position-dependent periodic external disturbance has been estimated pretty well after \( 4\pi \) radians. From Fig. 6, the control signal is reasonable.

**VI. CONCLUDING REMARKS**

In this paper, we have developed an adaptive compensator for the compensation of position-dependent disturbances and illustrated the ideas in the simulation of a roller coaster system. Our results show that state-dependent external disturbances in repetitive trajectory systems can be completely compensated for by using a state-dependent periodic adaptive compensator. We summarize the results as:

- The unknown state-periodic disturbance could be any kind of nonlinear function as long as the Lipschitz condition is satisfied.
- Lyapunov stability analysis was carried out in the state-domain.

The work considered in this paper is different from previous work in the following points:

- The state-periodic adaptive controller was established to compensate for an external state-periodic disturbance asymptotically.
- The precision tracking of the desired speed trajectory is achieved by using the proposed state-periodic adaptive compensator.
- To show the practical importance of the theory proposed in this paper, a state-dependent disturbance acting on a roller coaster system was compensated for along the state-domain.

It is believed that the suggested compensation method can be used in various practical applications such as servomotor control, tire-speed control, factory process control, robotic-manipulator, coordinate formation control, etc. As our future works, we would like to relax Assumption 3.2.
so that we could consider the case of $v(t) = 0$.

REFERENCES


