Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality

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Dedicated to Professor H.M. Srivastava on the occasion of his 65th birthday

Abstract

This paper provides a new analytical robust stability checking method of fractional-order linear time invariant interval uncertain system. This paper continues the authors’ previous work [YangQuan Chen, Hyo-Sung Ahn, I. Podlubny, Robust stability check of fractional-order linear time invariant systems with interval uncertainties, in: Proceedings of the IEEE Conference on Mechatronics and Automation, Niagara Falls, Canada, July, 2005, pp. 210–215] where matrix perturbation theory was used. For the new robust stability checking, Lyapunov inequality is utilized for finding the maximum eigenvalue of a Hermitian matrix. Through numerical examples, the usefulness and the effectiveness of the newly proposed method are verified.

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1. Introduction

After Newton and Leibniz discovered calculus in the 17th century, fractional-order calculus has been studied as an alternative calculus in mathematics [7]. As claimed in [6], fractional-order calculus will play an important role in mechatronic and biological systems. Recently, in control society, fractional-order dynamic systems and controls have gained an increasing attention [11,29,24,25,30]. Pioneering works in applying fractional calculus in dynamic systems and controls and the recent developments can be found in [13,14,23,1,34,12,22]. For more detailed explanation about the fractional dynamics and control, refer to [35].

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With regard to the stability of fractional-order linear time invariant (FO-LTI) system, there are numerous literature sources \([31,33,32,15,17,18,16,2,19]\), where internal stability, input–output stability, controllability, and observability have been studied. For interval FO-LTI systems, the first result on stability was discussed in \([26,27]\). Recently, the controllability issue for interval FO-LTI systems has been addressed for the first time in \([5]\). However, few result is available for the analytical robust stability condition of FO-LTI system with interval uncertainty. This paper focuses on the analytical robust stability problem of FO-LTI interval system with less conservatism.

There are actually some existing results for the robust stability of FO-LTI interval uncertain system as presented in \([4]\), where the matrix perturbation theory was used to find the ranges of interval eigenvalues. However, as commented in the same paper, the result could be conservative, because the suggested method calculates the ranges of eigenvalues in real part and imaginary part separately. Furthermore, if there is a big interval uncertainty in the nominal FO-LTI system, the suggested method in \([4]\) may not be suitable. Also recently, in \([21]\), LMI inequalities were proposed for stability checking of FO-LTI system. However, there was no discussion about the robustness of the FO-LTI system even though it provides some potential applications for the uncertain FO-LTI system. In this paper, it will be shown that the robust stability of some specific FO-LTI system with fractional commensurate order of \(1 \leq \alpha < 2\) can be effectively checked regardless of the interval perturbation amount. Note that, in \([21]\), for the stability of such a FO-LTI system with fractional commensurate order of \(1 \leq \alpha < 2\), an LMI was used for nominal stability checking.

In the following section, we will use Lyapunov inequality for \(D\)-stability condition \([8]\) for our main result. In Section 2, some basic background materials are given for extensive use in Section 3 where the main results are presented. In Section 4, illustrative examples are given. Conclusions will be given in Section 5 and possible future research items are given in Section 6.

2. FO-LTI interval system

In this study, we adopt the following Caputo definition for fractional derivative, which allows utilization of initial values of classical integer-order derivatives with known physical interpretations \([3,28]\)

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t f^{(n)}(\tau)(t-\tau)^{\alpha-n-1}d\tau,
\]

where \(n\) is an integer satisfying \(n - 1 < \alpha \leq n\).

Let us consider the following interval FO-LTI system:

\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t),
\]

where \(\alpha\) is the fractional commensurate order; the system matrix is uncertain in the sense that \(A \in \mathcal{A}^I = [\mathcal{A}, \bar{A}]\) \((\mathcal{A} \in \mathbb{R}^{n \times n})\), where \(\mathcal{A}\) and \(\bar{A}\) are lower/upper boundaries of the uncertain \(A\) in elementwise sense, and \(B \in \mathcal{B}^I = [\mathcal{B}, \bar{B}]\). Although \(B\) is uncertain, since we are interested in internal stability \([31]\), the robust stability problem considered in this paper is mainly related with \(x\) and \(A^I\). The robust stability problem of \(0 < \alpha < 1\) was studied in \([4]\). So, this paper focuses on the robust stability of \(1 \leq \alpha < 2\).

As shown in \([21]\), with \(1 \leq \alpha < 2\), when \(A\) matrix is deterministic without uncertainty, the stability condition for \(\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)\) is clearly

\[
\min_i |\arg(\lambda_i(A))| > \alpha \pi / 2, \quad i = 1, 2, \ldots, N.
\]

Thus, the robust stability test task for FO-LTI interval systems amounts to examining if

\[
\min_i |\arg(\lambda_i(A))| > \alpha \pi / 2, \quad i = 1, 2, \ldots, N \quad \forall A \in A^I.
\]

Fig. 1 shows the stability domain of an FO-LTI system with \(1 \leq \alpha < 2\). Hence, if all eigenvalues of \(A, A \in A^I\) are within inside this stable area, the interval FO-LTI system considered in this paper is then robustly stable. Therefore, the robust stability problem of FO-LTI interval system with \(1 \leq \alpha < 2\) is equivalent to checking if
all eigenvalues of $A \in A^I$ are within the stable area or not as shown in Fig. 1. In this paper, we propose using Lyapunov inequality for this test, which will be explained in the following section in detail.

3. New robust stability condition of FO-LTI interval system

It is a well-known fact [20] that if there exist positive definite Hermitian matrices $P > 0$ and $Q > 0$ such that

$$
\beta PA + \beta^T A^T P = -Q,
$$

(5)

where $\beta = \eta + j\zeta$ with $\tan(\pi/2 - \theta) = \eta/\zeta$, then all eigenvalues of $A$ are within the stable area of Fig. 1. Hence, if there is no interval model uncertainty, checking the existence of $P$ and $Q$ in (5) provides the stability of FO-LTI system. However, in this paper, we consider the robust stability with interval uncertainty. So, the solution is not straightforward. In this section, we develop an algorithm for checking this robust stability. For convenience, let us replace $\beta A = \hat{A}$. Then, $\hat{A}$ is a complex interval matrix. Simply, let us select $P = I$. Then,

$$
\hat{A} + \hat{A}^H = -Q
$$

(6)

where $\hat{A}^H$ is the transposed complex conjugate matrix of $\hat{A}$. So, if there exists $Q$ such that $Q = Q^H > 0$ for all $A \in A^I$, then our original FO-LTI interval system is robustly stable. Here, noticing $\hat{A} + \hat{A}^H$ is a interval Hermitian matrix, since eigenvalues of $\hat{A} + \hat{A}^H$ are real, we can conclude that if the maximum eigenvalue of $\hat{A} + \hat{A}^H$ is less than zero, the system is then robustly stable. That is, since it is certain that with interval uncertainty, $\hat{A} + \hat{A}^H$ is still Hermitian, if we can find the maximum eigenvalue of a Hermitian interval matrix, we can then estimate the robust stability of the FO-LTI interval system. For simplicity, we use the notation $A + A^H = T$. Then, we have two problems. The first one is to find the lower and upper boundary of interval Hermitian $T^4$ from $\beta$ and $A \in A^I$, and the second one is to find the maximum eigenvalue of $T \in T^4$.

For calculating $T^4$, from $T = (\eta + j\zeta)A + (\eta - j\zeta)A^T = \eta(A + A^T) + j\zeta(A - A^T)$, by calculating the extreme boundaries of the lower triangular part of $A + A^T$, $A \in A^I$ and the extreme boundaries of the lower triangular part of $A - A^T$, $A \in A^I$, we can find the lower and upper boundaries of $A + A^T$ and $A - A^T$. This process is straightforward by using interval computation arithmetics given in [9]. However, if we directly use interval software, the result could be very conservative, because in $\eta(A + A^T) + j\zeta(A - A^T)$, $A + A^T$ and $A - A^T$ are dependent each other. In other words, if we use interval computation software, the dependency between $A + A^T$ and $A - A^T$ will be ignored, which could result in certain conservatism. In the sequel, it will be shown that, actually, the above two problems can be combined together and vertex matrices of $A \in A^I$ could be used for calculating the maximum eigenvalue of $(\eta + j\zeta)A + (\eta - j\zeta)A^T$. For the explicit delivery of the idea, the following theorem is developed.
**Theorem 1.** Defining vertex matrices of $A \in A^1$ such as

$$A^v = \{ A: A = [\alpha^v_{ij}], \alpha^v_{ij} \in \{ \overline{\alpha_{ij}}, \underline{\alpha_{ij}} \} \},$$

the maximum eigenvalue of $\overline{A} + \overline{A} = T \in T^1$ occurs at one of vertex matrices of $A \in A^1$.

**Proof.** For the proof, we adopt an idea from [10]. Since $T$ is Hermitian and $T^4$ is interval Hermitian matrix, as explained in [10], the maximum eigenvalue of $T^4$ is calculated as

$$\overline{\lambda} = \max_{T \in T^4} \left( \max_{|x| = 1} x^T T x \right).$$  \hfill (7)

Following the procedure in [10], we denote the eigenvector as $x = u + jv$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$. Now, inserting $x = u + jv$ and $T = \eta(A + A^T) + j \zeta(A - A^T)$ into the quadratic form $x^T T x$, and noticing the diagonal terms of $A - A^T$ are zero, we have

$$x^T T x = \eta(a_{11} + a_{11})u_1^2 + \sum_{k=2}^{n} \eta(\alpha_{kk} + \alpha_{kk}) (u_k^2 + v_k^2) + \sum_{l=2}^{n} 2[\eta(\alpha_{1l} + \alpha_{1l}) u_l u_l + \zeta(\alpha_{1l} - \alpha_{1l}) u_l v_l]$$

$$+ \sum_{k=2}^{n} \sum_{l=k+1}^{n} 2[\eta(\alpha_{kl} + \alpha_{kl}) (u_k u_l + v_k v_l) - \zeta(\alpha_{kl} - \alpha_{kl}) (-u_k v_l + v_k u_l)].$$ \hfill (8)

Let us change the above equation as

$$x^T T x = 2\eta a_{11} u_1^2 + \sum_{k=2}^{n} 2\eta \alpha_{kk} (u_k^2 + v_k^2) + \sum_{l=2}^{n} 2[\eta(\alpha_{1l} + \zeta) u_l u_l + \alpha_{1l}(\eta u_l - \zeta u_l) v_l]$$

$$+ \sum_{k=2}^{n} \sum_{l=k+1}^{n} 2[\alpha_{kl}(\eta u_k u_l + v_k v_l) - \zeta(-u_k v_l + v_k u_l)] + a_{kk}(\eta u_k u_l + v_k v_l) + \zeta(-u_k v_l + v_k u_l)].$$ \hfill (9)

Now the following argument is straightforward. If $\eta \geq 0$, then since $u_1^2 > 0$ and $(u_k^2 + v_k^2) > 0$, $\overline{\lambda}$ occurs at $\overline{\alpha}_{kk}$. Otherwise, if $\eta < 0$, the $\overline{\lambda}$ occurs at $\overline{\alpha}_{kk}$. Similarly, if $(\eta u_l + \zeta u_l) \geq 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{ll}$ $(l > 1)$; if $(\eta u_l + \zeta u_l) \leq 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{ll}$ $(l \leq 1)$; if $(\eta(u_k u_l + v_k v_l) - \zeta(-u_k v_l + v_k u_l)) \geq 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{kl}$ $(k \neq l)$; and if $(\eta(u_k u_l + v_k v_l) - \zeta(-u_k v_l + v_k u_l)) \leq 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{kl}$ $(k \neq l)$. Also, if $(\eta u_k u_l + \zeta u_l) < 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{kl}$ $(l > 1)$; if $(\eta u_l + \zeta u_l) < 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{kl}$ $(l \leq 1)$; if $(\eta(u_k u_l + v_k v_l) - \zeta(-u_k v_l + v_k u_l)) < 0$, then $\overline{\lambda}$ occurs at $\overline{\alpha}_{kl}$ $(k \neq l)$. This completes the proof. \hfill \Box

Usually, since the sign of $\eta$ is known, the diagonal terms of $A \in A^1$ can be selected according to the sign of $\eta$. That is, if $\eta \geq 0$, for $\overline{\lambda}$, we select $\overline{\alpha}_{kk}$. Otherwise, if $\eta < 0$, for $\overline{\lambda}$, we select $\overline{\alpha}_{kk}$. This will save the computational cost. For $\underline{\lambda}$, we can also develop a similar theorem such as

**Theorem 2.** The minimum eigenvalue of $\overline{A} + \overline{A} = T \in T^1$ occurs at one of vertex matrices of $A \in A^1$.

**Proof.** Repeating the above discussions, the proof is straightforward. \hfill \Box

For reducing the computational amount, if $\eta \geq 0$, for $\overline{\lambda}$, we select $\overline{\alpha}_{kk}$; otherwise, if $\eta < 0$, for $\overline{\lambda}$, we can select $\overline{\alpha}_{kk}$.

4. Two illustrative examples

4.1. Example-1

Let us check the robust stability of the following fractional interval system:

$$\frac{d^2 x(t)}{dt^2} = A^1 x,$$ \hfill (10)
where $x = 1.5$, and $A^I \in \mathcal{A}^I = [A, \overline{A}]$ with

$$
A = \begin{pmatrix}
-1.8 & 0.4 & 0.8 \\
-1.2 & -3.6 & 0.8 \\
-0.6 & -1.8 & -3.0
\end{pmatrix},
$$

$$
\overline{A} = \begin{pmatrix}
-1.2 & 0.6 & 1.2 \\
-0.8 & -2.4 & 1.2 \\
-0.4 & -1.2 & -2.0
\end{pmatrix}.
$$

Since $\theta = \pi/4$, we have $\eta = 1$ and $\zeta = 1$. Using Theorem 1, we found $\overline{\alpha} = -0.40$. So, the FO-LTI interval system is robustly stable. For the verification of the analytical result given in the above, we also performed a random test. Fig. 2 shows the random test result. Eigenvalues of all random plants, which are within interval uncertainty, are calculated. Fig. 2 shows that all eigenvalues are within stable area. So, the FO-LTI interval system in this example is robustly stable.

### 4.2. Example-2

Next, we check the robust stability of another fractional interval system given in the following:

$$
\frac{d^\alpha x(t)}{dt^\alpha} = A^I x,
$$

(11)

where $x = 1.5$, and $A^I \in \mathcal{A}^I = [A, \overline{A}]$ with

$$
A = \begin{pmatrix}
-1.95 & 0.35 & 0.7 \\
-1.3 & -3.9 & 0.7 \\
-0.65 & -1.95 & -3.25
\end{pmatrix},
$$

$$
\overline{A} = \begin{pmatrix}
-1.05 & 0.65 & 1.3 \\
-0.7 & -2.1 & 1.3 \\
-0.35 & -1.05 & -1.75
\end{pmatrix}.
$$

![Fig. 2. Eigenvalues of FO-LTI within interval uncertainty of Example-1.](image-url)
Since $\theta = \pi/4$, we have $\eta = 1$ and $\zeta = 1$. Using Theorem 1, we found $\lambda = 0.3747$, so the FO-LTI interval system is robustly unstable. Fig. 3 shows the random test. Eigenvalues are calculated for random plants which are within lower and upper boundaries of the interval plant. We observe that some of eigenvalues are outside of the stable area. So, the FO-LTI interval system of Example-2 is considered as robustly unstable, which agrees with the analytical solution.

5. Conclusions

In this paper, we provided a new analytical method for checking the robust stability of FO-LTI interval system with fractional commensurate order $1 < \alpha < 2$. The motivation of this work is to overcome some drawbacks of the authors' early work [4] where the matrix perturbation theory was used. From the numerical tests, we have found that the newly suggested method finds the robust stability very accurately.

6. Concluding remarks

This paper proposed a new robust stability checking method for a class of FO-LTI interval systems with the commensurate order limited to $1 < \alpha < 2$. We have found that it is difficult to apply Lyapunov inequality to the FO system with commensurate order $0 < \alpha < 1$, because there is no robust $D$-stability result for this. In our future research efforts, we wish to develop an analytical Lyapunov inequality for the FO-LTI system with fractional commensurate order $0 < \alpha < 1$.

Another interesting research topic is in solving (5) such as

$$\beta PA + \beta^r A^T P = -Q \quad \forall A \in A^1. \quad (12)$$

In this paper, we selected $P = I$ for the easy derivation of the analytical solution. However, it will be also interesting to select $P = P^*$ that is a solution for the nominal $A$ such as

$$\beta PA + \beta^r (A)^T P = -I, \quad A = A^0. \quad (13)$$
Then, by solving the quadratic form as done in this paper, we may find the maximum eigenvalue of $\beta P A + \beta^*(A)^T P$, which is still Hermitian, at one of the vertex matrices of $A \in A^I$. This result will be reported in the near future.

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