1. Context

Fractional calculus is about the integration or differentiation of non-integer orders. The use of “fractional” is purely due to historical reasons [1]. Using fractional order differential equations is believed to be able to better characterize the nature around us. Using an integer order model is only for our own convenience. Depending on the scale on which we characterize the dynamics of a system, more and more evidences are found that using fractional order model is ubiquitous and unavoidable [1, 2, 3].

For dynamic systems and controls, there are four possibilities 1) Integer-order plant model with integer order controller; 2) Integer-order plant model with fractional order controller; 3) Fractional-order plant with integer-order controller; and 4) Fractional-order plant model with fractional-order controller [2]. Case-1 is our traditional (integer-order) control. An example of Case 2) can be found in [3] and an example of Case-4 can be found in [4]. Examples for Case-3 are abundant such as conventional integer-order controls for flexible structures whose transfer functions are irrational or of fractional order.

It is interesting to note that, reaction curve based rough modeling technique, although widely used in process industry to get a first order plus delay time (FOPDT) for initial tuning of a working PID controller with techniques such as Ziegler-Nichols tuning rule, has not been put enough emphasis in control textbooks. We believe that, every practicing control engineer and control engineering educator should pay attention to the importance of reaction curve based modeling and the controller design techniques based on the model obtained this way.

The current paper under discussion goes a step further to suggest that, with reaction curve of a special S-shape, it might be beneficial to build a four-parameter fractional order of the form

\[ \hat{G}(s) = \frac{K}{1 + Ts^\alpha} e^{-Ls}. \]

It is an extended form of the FOPDT where \( \alpha = 1 \). This is a meaningful extension. A further extension in the paper is to introduce the so-called “five-parameter” model with \( r \) an integer

\[ \hat{G}(s) = \frac{K}{(1 + Ts^\alpha)^r} e^{-Ls}. \]

2. Model Parameter Retrieving Procedures

The main contribution of the paper is on how to retrieve the parameters \( K, T, L \) and also \( \alpha \). The starting point is \( \hat{g}(t) \), the open-loop step response of \( \hat{G}(s) \) which is written in the following form

\[ \hat{g}(t) = K \left( 1 - E_{\alpha,1} \left( \frac{-1}{T} (t-L)^\alpha \right) \right) u(t-L) \]

using Mittag-Leffler function in two parameters \( E_{\alpha,1} \). For “five-parameter” model, its impulse response is shown to be

\[ h(t) = \frac{T^{-r}}{(r-1)!} \int_{t}^{\infty} d^{r-1} \frac{d^{(r-1)}}{dt^{(r-1)}} \left( E_{\alpha,r} \left( -t^\alpha / T \right) \right) \]
which is also based on Mittag-Leffler function in two parameters. However, as shown in page 224 of [2], it holds that

$$
\mathcal{L}^{-1}\left[\frac{s^{\alpha \gamma - \beta}}{(\alpha a + a)^\alpha}\right] = t^{\beta - 1} E_{\alpha, \beta}^{\gamma}(-at^\alpha)
$$

where $\alpha$, $\beta$, $\gamma$ and $a$ are real numbers. It is more compact to use Mittag-Leffler function in three parameters or generalized Mittag-Leffler function to get the impulse response of “five-parameter” model with the Matlab code in [2]. Note that $r$ here can be a non-integer.

With the above analytical expressions, it is not hard to establish procedures for retrieving the parameters for “four-parameter” model as shown in the paper. However, for “five-parameter” model, $r$ must be an integer in the authors’ work.

3. Model-Based Controller Designs

The authors discussed several model-based design methods, all resulting “fractional order controllers” for fractional-order plants, which falls in the Case-4 in Section 1. These techniques are

1. Internal Model Control (IMC) Method
2. Fractionalized Smith Predictor Based Controllers
3. Fractionalized Haalman Method
4. $\lambda - \gamma$ Tuning Method.

While details can be found in the paper, we noted that robust stability is not addressed explicitly except the delay uncertainty considered in Section 5.4. Stability margins such as gain margin and phase margin are not discussed neither. Performance in time and/or frequency domain was not pre-specified.

4. Summary and Further Remarks

In summary, this paper presented an interesting fractional order extension of S-shape reaction curve based models. Based on these fractional order models retrieved from the S-shape reaction curve, four fractional order control designs are presented and illustrated. This work will motivate more related investigations, in particular the fractional order controller design and tuning methods based on the “four-parameter” and “five-parameter” model classes. Specifically, we offer the following remarks for possible further studies:

- For “five-parameter” model, it is not necessary to limit $r$ to be an integer. How to reliably determine $r$ is an interesting topic and it is not sure if S-shape reaction curve provide enough information to retrieve the $r$.
- Noise in the S-shape reaction curve measurement should be discussed and the sensitivity of the retrieved parameters with respect to the noise level should be considered, too. For example, which parameter is least reliably retrievable?
- Model-based controller designs as presented in the current paper would be of more practical use if the control designs could begin with performance specifications and constraints dictated by practice.
- The difference between the ideal fractional order controller designed and the approximated version, e.g., the approximated high order integer-order transfer function, should be illustrated and even compensated if necessary, in practical applications. However these approximation error affects the expected performance is still an open problem.

References

Discussion on: “Simple Fractional Order Model Structures and their Applications in Control System Design”

Ivo Petráš
Faculty of BERG, Technical University of Košice, Slovak Republic

The paper by M. Tavakoli-Kakhki, M. Haeri, and M. S. Tavazoei presents a four-parameter and a five-parameter model structure to characterize the dynamic response of a system possessing an S-shaped step response. Different strategies were proposed for estimating the parameters of the approximating models. For these simple model structures new control design methods were introduced and some existing methods were modified. Finally, some numerical examples were provided to show the applicability of the proposed procedures.

There are two basic ideas of this paper. The first one is represented by three different strategies in order to determine the parameters of the proposed model (four or five-parameter structure). The second one is represented by control design approaches which are applied to proposed models. They can be summarized as follows:

- Internal Model Control Method
- Fractionalized Smith Predictor Based Controllers
- Fractionalized Haalman Method
- λ − γ Tuning Method

First, I would say that these four and five-parameter models may cover only certain class of fractional-order systems. Second, the control design procedures were limited only to the above-listed procedures, without additional well-known methods, in order to compare them with each other.

This discussion is focused on two issues. The first one considers the possibility of using a fractional-order 1\(^{\mu}\)D\(^{\delta}\) controller for the four-parameter model. Second, a simple numerical method for simulation of the fractional-order system is outlined.

To extend this work, let us consider the four-parameter model represented by the transfer function:

\[
G(s) = \frac{K}{Ts^\alpha + 1} e^{-Ls},
\]

(1)

where \(K\) is the static gain, \(L\) is the time-delay, \(T\) is the time constant of the system and \(\alpha\) is the fractional order such that \(0 < \alpha < 2\). The fractional-order 1\(^{\mu}\)D\(^{\delta}\) controller is represented by the transfer function:

\[
C(s) = K_i \left( \frac{1}{s^\mu} + Ts^{\delta} \right),
\]

where \(K_i\) is an integration constant and the derivative order \(\delta = \alpha - \mu\). The open loop transfer function is given by [1]:

\[
C(s)G(s) = K_i K e^{-Ls} s^{\mu}, \quad 1 < \mu < 2.
\]

Let us consider that user-defined gain margin \(A_m\) and phase margin \(\phi_m\) are defined as:

\[
\begin{align*}
|C(j\omega_p)G(j\omega_p)| &= \frac{1}{A_m} \\
\phi_m &= \pi + \angle C(j\omega_g)G(j\omega_g),
\end{align*}
\]

where \(\omega_p\) and \(\omega_g\) are the phase and gain crossover frequencies of the open loop system.

On substitution, the gain relation is:

\[
A_m = \frac{\omega_p^\mu}{K_i K}
\]

and the phase relation is:

\[
\phi_m = \pi - L\omega_g - \frac{\mu \pi}{2}.
\]

The relation between \(\phi_m\) and \(A_m\) depends on \(\mu\) is given

\[
A_m = \left( \frac{\pi - \mu \pi}{\pi - \phi_m - \mu \pi} \right)^\mu.
\]

(2)

Note that there exist \(\mu\) in restricted area (\(\mu < 1\) or \(\mu > 2\)) for given desired \(\phi_m\) and \(A_m\). Therefore to select \(\phi_m\) and \(A_m\) we have to use suggestions and limitations described in [1]. For example, if \(\phi_m = 1.0472\) (60\(^{\circ}\)) and \(A_m = 1000\), then Eq. (2) has solution for \(\mu \approx 1.33\).
Once the value \( \mu \) is obtained, the corresponding values of \((\omega_p, \omega_g, K_i)\) are computed as:

\[
\begin{align*}
\omega_p &= \frac{\pi - \mu \pi}{L} \\
\omega_g &= \frac{\pi - \phi_m - \mu \pi}{L} \\
K_i &= \frac{\omega_p}{K_A m} = \frac{\omega_g}{K}
\end{align*}
\]

Thus, for given four-parameter model (1), desired gain margin \( A_m \), phase margin \( \phi_m \) and given \( \mu \), we obtain parameters of the fractional-order \( I^\mu D^\delta \) controller represented by the transfer function:

\[
C(s) = \frac{K_i}{s^\mu} + K_d s^\delta,
\]

where \( K_d = K_i T \) and \( K_i \) is given in (3).

The numerical method (predictor-corrector) and rational approximation method used for computing unit step responses and other simulations are the appropriate methods. However, in such simple examples a method proposed in [2] or a “Grünwald-Letnikov” method [3] with a “short memory principle” can be used as well with very good results. It was confirmed via comparison of all of them each other on the simulation example in [3].

Generally, for a simple fractional-order differential equation

\[
_0D^\alpha_t y(t) = f(y(t), t),
\]

a numerical solution can be expressed as

\[
y(t_k) = f(y(t_k), t_k)h^\alpha - \sum_{j=0}^{k} c_j^{(\alpha)} y(t_{k-j}),
\]

where \( t_k = kh, \) \( h \) is time step, \( \nu \) will be \( \nu = 1 \) for \( k < \frac{(L_m/\gamma)}{h} \) and \( \nu = k-(L_m/\gamma) \) for \( k > \frac{(L_m/\gamma)}{h} \), with “memory length” \( L_m \), or without using the “short memory principle”, we put \( \nu = 1 \) for all \( k \). Binomial coefficients \( c_j^{(\alpha)} \) are calculated according to the relation

\[
c_j^{(\alpha)} = \left( 1 - \frac{1 + \alpha}{j} \right) c_{j-1}^{(\alpha)}, \quad c_0^{(\alpha)} = 1.
\]

References


Final Comments by the Authors

Mahsan Tavakoli-Kakhki, Mohammad Haeri, Mohammad Saleh Tavazoei

Advanced Control System Lab, Electrical Engineering Department, Sharif University of Technology, Tehran, Iran

In the paper “Simple Fractional Order Model Structures and Their Applications in Control System Design”, some studies have been done as a pioneering work in the subject of extracting simple fractional order models from the system step tests. In this work, some strategies have been suggested for estimating the parameters of simple four-parameter and five-parameter fractional order models describing the behavior of those systems having S-shaped step responses. Also, different control techniques have been discussed to show some applications of the proposed simple structure models in control system design. As discussed by Chen, Luo, and Petras, there are some interesting topics which invite future researches in the continuation of this work. Some of these topics are listed in the following three categories:

1. Model Extraction

This category can include the following research topics: Proposing other simple fractional order models for describing dynamical systems (for example models describing systems with non S-shape step responses), Suggesting appropriate strategies for determining the parameters of these new simple models, and studying on the noise effect on the reliability of the determined parameters.
2. Control Applications

Applying the proposed simple fractional order models in control system design based on satisfying some frequency response or time response performance specifications, and studying on the robustness of the designed model-based controllers in the presence of uncertainty in the model parameters are placed in this category.

3. Practical Implementation

Finding appropriate methods for implementing the obtained fractional order controllers would be of practical worth. Also, investigating the differences between the original designed fractional order controller and its implemented counterpart, and trying to reduce the effects of these differences helps the engineers to achieve the intended control objectives more effectively. It is worth mentioning that some preliminary works on this subject in the viewpoint of stability preservation problem have been previously done in [1]–[2].

References